A New Discontinuous Galerkin Formulation for the Boussinesq system with Navier-type boundary condition

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Abstract

In this work we propose and analyze a discontinuous Galerkin method for the nonlinear coupled Navier–Stokes/temperature (or Boussinesq) equations with Navier-type boundary condition for the velocity. Existence and uniqueness of the solution are obtained under a small data condition. We provide *a priori* error estimates in terms of a natural energy norms for the velocity, the pressure and the temperature. To our knowledge, it is the first time that a discontinuous Galerkin approximation for the full nonlinear coupled Bousinessq system, with Navier-type boundary condition for the velocity, is proposed and completely analyzed in both, continuous and discrete settings.

Key words: Stationary Boussinesq equations; discontinuous Galerkin method; Navier boundary condition; fixed point theory, existence, *a priori* error estimate, discrete inequality.

Mathematics subject classifications (2020): 65N12, 65N15, 65N22 65N30, 35Q30, 35Q79.

1 Introduction

Over the past few decades, the community of numerical analysts of partial differential equations has extensively focused on developing precise and efficient new finite element methods for the Boussinesq problem. This model is well-known for its relevance in engineering sciences as it deals with fluid motion generated by density differences caused by temperature gradients. Mathematically, it is a nonlinear system of PDEs which composed of the stationary incompressible Navier–Stokes equations for the velocity and pressure variables and a convection–diffusion equation for the temperature variable. Such coupling is done by means of a buoyancy term in the Navier-Stokes euations depending on the temperature and convective term in the heat equation depending on the velocity of the fluid. The Boussinesq approximation was justified and used to study several applications such as in geophysics to model climate prediction [12], oceanography for studying oceanic flows [20, 22] and also in magnetohydrodynamic flows, see [10]. Moreover, simulations that combine fluid flow and temperature are invaluable in predicting the performance of physical designs across a wide range of engineering applications. By conducting these simulations before the manufacturing process, it becomes possible to significantly reduce the costs associated with developing new products [27, 29].

The Boussinesq system has been the topic of multiple independent studies. For the numerical front, several discretizations have been employed to solve this system with Dirichlet boundary condition for the fluid. Let us describe some of these contributions. Bernardi et al. [3], proved existence and uniqueness of the weak solution for the Boussinesq system and proposed a finite element approximation. Farhfoul et al. [13] proposed a novel mixed formulation for the two-dimensional Boussinesq

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where the existence of solution and convergence of the numerical scheme have been proved near a non-singular solution. Colmenares et al. [24] analyzed a new finite element method with exactly divergence-free velocities for the numerical simulation of a generalized Boussinesq problem where the viscosity and the thermal conductivity depend on the temperature of the fluid. More recently, Colmenares et al. proposed another new divergence-Conforming DG-Mixed Finite Element Method for the Stationary Boussinesq Problem [23].

However, it has been showed that the Dirichlet boundary condition for the fluid is inadequate in some situations such as in the mechanics of thin films, multiple interfaces problems, the flow of rarefied fluids, the flow of a fluid in perforated domains, flow of blood through blood vessels (see e.g. [19]). In these cases, the French engineering Claude Navier ([21]) proposed that the tangential velocity should be proportional to the tangential stress on the boundary together with with the impermeability condition $\boldsymbol{u}\cdot\boldsymbol{n}=0$. These condition are known as the Navier boundary condition. A generalization of the Navier condition can be written as:

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{and} \quad 2[D(\boldsymbol{u})\boldsymbol{n}]_\tau+\alpha\boldsymbol{u}_\tau=\boldsymbol{0} \quad \text{on } \Gamma,$$

where $\alpha\geq 0$ is the coefficient of friction and

$$D(\boldsymbol{u})_{ij}=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big),\ 1\leqslant i,j\leqslant 3.$$

The subscript τ denotes the vectorial tangential trace of any vector, defined by $\boldsymbol{v}_\tau := \boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$. Note that, when α tends to infinity, we formally recover the Dirichlet conditions. In the case of a flat boundary and zero friction coefficient α, this Navier condition is equivalent to the following Navier-type conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=0 \quad \text{on } \Gamma, \tag{1.1}$$

Concerning the study of Boussinesq system with the above boundary conditions, the literature is rather scarce. The idea is originally described in [1], where the steady Boussinesq problem with Navier boundary condition (1.1) have investigated and the existence of weak solutions in the Hilbertian case and the L^p-regularity of these weak solutions are proved. Regarding the devising of suitable numerical methods, to the best of our knowledge, there is no work in literature which approximate Boussinesq system with Navier type Boundary condition (1.1).

In the current paper, we propose and analyze an interior penalty discontinuous Galerkin method for the Boussinesq model with Navier-type boundary conditions (1.1) for the velocity and homogeneous boundary condition for the temperature. More precisely, given a fluid occupying the region Ω, a force per unit mass $\boldsymbol{g}$ and a heat source f, the model of interest reads: Find a velocity field $\boldsymbol{u}$, a pressure field p and a temperature field θ such that:

$$-\nu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}+\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u}+\nabla p-\theta\boldsymbol{g}=0 \qquad \text{in}\,\Omega, \tag{1.2a}$$

$$\operatorname{div}\boldsymbol{u}=0, \qquad \text{in}\,\Omega, \tag{1.2b}$$

$$-\kappa\,\Delta\theta+\boldsymbol{u}\cdot\nabla\theta=f \qquad \text{in}\,\Omega, \tag{1.2c}$$

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=0 \qquad \text{on } \Gamma, \tag{1.2d}$$

$$\theta=0 \qquad \text{on } \Gamma, \tag{1.2e}$$

Here $\Omega\subset\mathbb{R}^3$ is an open convex domain. ν is the kinematic viscosity of the fluid, κ is the thermal diffusivity of the fluid.

DG method is a type of finite element method where the solution is approximated by a piecewise polynomial function that is discontinuous across element boundaries. These methods are attractive because they are element-wise conservative, easily implementable on unstructured meshes and they are high-order methods.

The main difficulty is to prove the stability of the bilinear forms arising from the Navier-Stokes convection O_h (see definition in (3.3)) and the coupling term $\mathcal{C}_h$ (see definition in (3.5)) terms on the discrete spaces.

To give the reader an idea of the main features of our DG method, let us compare it with other discontinuous Galerkin methods that have been devised for Bousinessq system or related problems.

In a recent work [9], the authors proposed an analysis of a DG scheme for the Boussinesq system where the boundary conditions are of Dirichlet type for the velocity field and of mixed type for the temperature. The general case of a Lipschitz domain Ω is considered. Since the solution $\boldsymbol{u}$ (respectively θ) belongs to $\boldsymbol{H}^1(\Omega)$ (respectively to $H^1(\Omega)$) and hence to $\boldsymbol{L}^6(\Omega)$ (respectively to $L^6(\Omega)$) by Sobolev embedding, they show the continuity of the nonlinear terms by applying an L^2-L^4-L^4 argument. For this end, they apply the discrete inequality derived in [17, 18] (see also [11]) in two and three dimensional bounded domains with Lipschitz boundary., which states that the $\boldsymbol{L}^6$ norm is controlled by the following DG norm denoted by $\|\cdot\|_h$:

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^6(\Omega)} \leq C\,\|\boldsymbol{v}_h\|_h := C\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{v}_h\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h}\frac{\gamma}{h_e}\,\|[\![\boldsymbol{v}_h]\!]\|_{0,e}^2\Big)^{1/2}. \tag{1.3}$$

This approach is not suitable for the problem that we propose to study. Indeed, in the case of a non-smooth domain and given the boundary conditions considered for the velocity field $\boldsymbol{u}$, the solution $\boldsymbol{u}$ of the system (1.2) is only in $\boldsymbol{H}^{1/2}(\Omega)$ and hence in $\boldsymbol{L}^3(\Omega)$ by Sobolev embedding. Therefore, the previous argument can be again applied to show the continuity of the form $\mathcal{C}_h$ in the heat equations but cannot be applied to show the continuity of the nonlinear form O_h in the Navier-Stokes equations. Additional regularity assumptions on the exact solution are therefore necessary.

Moreover, due to the boundary conditions considered in this work for the velocity, the DG energy norm for $\boldsymbol{u}$ is different from that defined in (1.3). In [25,26], a discrete functional analysis is established and the authors show a new discrete inequality analogous to (1.3) on discontinuous spaces provided that the boundary of the domain is sufficiently regular. More precisely, for $1 \leq p \leq 6$, they show that the L^p norm is controlled by the following DG norm:

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^p(\Omega)}^2 \leq C|\!|\!|\boldsymbol{v}_h|\!|\!|^2 := C\sum_{T\in\mathcal{T}_h}\Big(\|\operatorname{div}\boldsymbol{v}_h\|_{0,T}^2+\|\mathbf{curl}\,\boldsymbol{v}_h\|_{0,T}^2\Big)+\sum_{e\in\mathcal{F}_h}\frac{\sigma}{h_e}\,\|[\![\boldsymbol{v}_h]\!]_N\|_{0,e}^2+\sum_{e\in\mathcal{F}_h^I}\frac{\sigma}{h_e}\,\|[\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2\,,$$

This new discrete inequality allows the application of an L^4-L^4-L^2 argument to show the continuity of the nonlinear forms associated with our DG scheme and then to prove the existence of a solution to the discrete problem.

The rest of the paper is structured as follows. In Section 2, we introduce and show the well-posedenss of the continuous variational problem by applying the Leray-Schauder's theorem. A discontinuous Galerkin formulation based on the classical interior penalty (IP) symmetric method is presented in Section 3 where the existence and uniqueness of approximate solutions are proved. In Section 4, we state and prove the convergence and our *a priori* error estimates for velocity, the pressure and temperature in energy norms.

2 Analysis of the continuous problem

2.1 Preliminaries and notations

Let Ω be an open, convex, and bounded domain in $\mathbb{R}^3$. For $p \in [1, +\infty]$, we denote by $L^p(\Omega)$ the usual Lebesgue space endowed with the norm $\|\cdot\|_{L^p(\Omega)}$ and $L_0^2(\Omega)$ is the space consisting of functions $q \in L^2(\Omega)$ that hold $\int_\Omega q = 0$. We adopt standard notation for the Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$.

In addition, according to the boundary conditions in (1.2d), and in view of deriving a weak formulation for (1.2), we introduce the following functional space:

$$\boldsymbol{H}_T^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on}\,\Gamma\} \tag{2.1}$$

We recall that the space $\boldsymbol{H}_T^1(\Omega)$ is continuously embedded in $\boldsymbol{H}^1(\Omega)$ if Ω is convex or the boundary of Ω is of class $\mathcal{C}^{1,1}$ (c.f. [16, Theorem 3.7] and [2, Theorem 3.4]). In addition we recall the following Poincaré inequalities, needed for proving the existence of a weak solution: there exist $C_P > 0$ and $\widetilde{C}_P > 0$ such that [16]

$$\|\phi\|_{H^1(\Omega)} \leq C_P \,\|\nabla\,\phi\|_{L^2(\Omega)}\,, \qquad \forall \phi \in H_0^1(\Omega), \tag{2.2a}$$

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq \widetilde{C}_P\big(\,\|\mathbf{curl}\ \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^2(\Omega)}\,\big), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_T^1(\Omega). \tag{2.2b}$$

We also recall that the space $\boldsymbol{H}^1(\Omega)$ is compactly imbedded in $\boldsymbol{L}^q(\Omega)$ for any exponent $1 \leq q \leq 6$ and we have

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \leq \delta\,\|\boldsymbol{v}\|_{1,\Omega}\,, \tag{2.3}$$

where δ is a positive constant depending on the domain Ω.

2.2 Weak Formulation

The weak formulation of (1.2) reads : Find $(\boldsymbol{u}, p, \theta) \in \boldsymbol{H}_T^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ such that

$$A(\boldsymbol{u},\boldsymbol{v}) + O(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + B(\boldsymbol{v},p) = D(\theta,\boldsymbol{v}), \tag{2.4a}$$

$$B(\boldsymbol{u},q) = 0 \tag{2.4b}$$

$$M(\theta,\psi) + \mathcal{C}(\boldsymbol{u};\theta,\psi) = F(\psi), \tag{2.4c}$$

for all $(\boldsymbol{v}, q, \psi) \in \boldsymbol{H}_T^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega)$, where the bilinear forms are defined by

$$\begin{aligned}
&A(\boldsymbol{u},\boldsymbol{v}) = \nu\Big(\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x} + \int_\Omega (\mathrm{div}\,\boldsymbol{u})\,(\mathrm{div}\,\boldsymbol{v})\,d\boldsymbol{x}\Big),\\
&M(\theta,\psi) = \kappa\int_\Omega \nabla\theta\cdot\nabla\psi\,d\boldsymbol{x}, \qquad D(\theta,\boldsymbol{v}) = \int_\Omega \theta\boldsymbol{g}\cdot\boldsymbol{v}\,d\boldsymbol{x}\\
&O(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) = \int_\Omega (\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{w})\cdot\boldsymbol{v}\,d\boldsymbol{x}, \qquad \mathcal{C}(\boldsymbol{u};\theta,\psi) = \int_\Omega (\boldsymbol{w}\cdot\nabla\theta)\psi\,d\boldsymbol{x},\\
&B(\boldsymbol{v},q) = -\int_\Omega (\mathrm{div}\,\boldsymbol{v})q\,d\boldsymbol{x}, \qquad F(\psi) = \int_\Omega f\psi\,d\boldsymbol{x}.
\end{aligned} \tag{2.5}$$

We describe a fixed point strategy that allows to solve the problem (2.4). First, we eliminate the pressure from the problem by restricting to the subspace

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{H}_T^1(\Omega); \quad B(\boldsymbol{u},q) = 0, \quad \forall q \in L_0^2(\Omega)\}$$

We then consider the equivalent problem : find $(\boldsymbol{u}, \theta) \in \boldsymbol{K} \times H_0^1(\Omega)$, such that

$$A(\boldsymbol{u},\boldsymbol{v}) + O(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = D(\theta,\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \tag{2.6a}$$

$$M(\theta,\psi) + \mathcal{C}(\boldsymbol{u},\theta,\psi) = F(\psi), \quad \forall \psi \in H_0^1(\Omega). \tag{2.6b}$$

The bilinear form B is continuous in $\boldsymbol{H}_T^1(\Omega) \times L_0^2(\Omega)$ and satisfies the following inf-sup condition: there exists a constant $\beta > 0$ such that

$$\sup_{\boldsymbol{v}\in\boldsymbol{H}_T^1(\Omega)} \frac{B(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \geqslant \beta\,\|q\|_{L^2(\Omega)}\,, \tag{2.7}$$

which is a direct consequence of its analogue with $\boldsymbol{H}^1_T(\Omega)$ replaced by $\boldsymbol{H}^1_0(\Omega)$ (see [4,5]).

As a consequence, it is easy to see that problems (2.4) and (2.6) are equivalent. In fact, we have the following standard result [16]

Lemma 2.1 *If $(\boldsymbol{u},\theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ is a solution of (2.6), then there exists $p \in L^2_0(\Omega)$ such that $(\boldsymbol{u},p,\theta)$ is a solution of (2.4). Conversely, if $(\boldsymbol{u},p,\theta) \in \boldsymbol{K} \times L^2_0(\Omega) \times H^1_0(\Omega)$ is a solution of (2.4), then $\boldsymbol{u} \in \boldsymbol{K}$ and $(\boldsymbol{u},\theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ is a solution of (2.6).*

We construct a map $\mathscr{S}$ by linearizing the term D and the convection terms O and $\mathcal{C}$ in problem (2.6) and show existence of a fixed point of $\mathscr{S}$. Define $\mathscr{S}$ as follows:

$$\begin{aligned} \mathscr{S} \colon \boldsymbol{K} \times H^1_0(\Omega) &\to \boldsymbol{K} \times H^1_0(\Omega) \\ (\boldsymbol{w},\phi) &\mapsto \mathscr{S}(\boldsymbol{w},\phi) = (\boldsymbol{u},\theta), \end{aligned} \tag{2.8}$$

where $(\boldsymbol{u},\theta)$ satisfies the linearized problem: Find $(\boldsymbol{u},\theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ such that

$$\begin{aligned} A(\boldsymbol{u},\boldsymbol{v}) + O(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) &= D(\phi,\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \\ M(\theta,\psi) + \mathcal{C}(\boldsymbol{w},\theta,\psi) &= F(\psi), \quad \forall \psi \in H^1_0(\Omega). \end{aligned} \tag{2.9}$$

Observe that, thanks to Lemma 2.1, a fixed point $(\boldsymbol{u},\theta)$ of the map $\mathscr{S}$ is the component of a solution $(\boldsymbol{u},p,\theta)$ of problem (2.4). The well posedness of (2.4) will be addressed in Subsection 2.4 by applying the Leray-Schauder theorem. It requires some stability properties of the forms defined above.

2.3 Stability properties of the forms

In this subsection, we show somme crucial stability properties of the forms that are used to prove the well posedenss of the problem (2.4). First, we discuss the coercivity properties of the forms A, M, O and $\mathcal{C}$. We have the following result.

Proposition 2.1 *There exist positive constants α_0 and α_1 depending on Ω such that*

$$A(\boldsymbol{v},\boldsymbol{v}) \geq \nu\alpha_{_A} \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)}, \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \tag{2.10}$$

$$M(\psi,\psi) \geq \kappa\alpha_{_M} \|\psi\|^2_{H^1(\Omega)}, \quad \forall \psi \in H^1_0(\Omega), \tag{2.11}$$

Furthermore, the following results hold:

(i) Let $\boldsymbol{w},\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. Then we have

$$O(\boldsymbol{w},\boldsymbol{v},\boldsymbol{v}) = 0. \tag{2.12}$$

(ii) Let $\boldsymbol{w} \in \boldsymbol{K}$ and $\phi \in H^1_0(\Omega)$. Then we have

$$\mathcal{C}(\boldsymbol{w},\psi,\psi) = 0 \tag{2.13}$$

Proof. The coercivity of the froms A and M follows from the Poincaré-Friedrichs inequalities (2.2a) and (2.2b). The property in (2.12) is obvious. It follows from the definition of O and the fact that the vector $\mathbf{curl}\,\boldsymbol{w} \times \boldsymbol{v}$ is orthogonal to $\boldsymbol{v}$. The property in (2.13) is a consequence of the following well-known skew-symmetry property of the form $\mathcal{C}$ with respect to the last two components:

$$\mathcal{C}(\boldsymbol{w},\theta,\psi) = -\mathcal{C}(\boldsymbol{w},\psi,\theta), \qquad \forall \boldsymbol{w} \in \boldsymbol{K}, \forall(\theta,\psi) \in H^1_0(\Omega) \times H^1_0(\Omega).$$

□

We also need to study the continuity of the forms involved in the variational formulation (2.4). We have the following results where the proof is obtained by applying Cauchy–Schwarz's inequality to the different forms.

Proposition 2.2 *There exist positive constants C_{A}, C_{M}, C_{B} and C_{F} depending on Ω such that*

$$\begin{aligned} A(\boldsymbol{u},\boldsymbol{v}) &\leq \nu C_{A} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, &(2.14)\\ M(\theta,\psi) &\leq \kappa C_{M} \|\theta\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}, &(2.15)\\ B(q,\boldsymbol{v}) &\leq C_{B} \|q\|_{L^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, &(2.16)\\ F(\psi) &\leq C_{F} \|f\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}, &(2.17)\end{aligned}$$

for all $\boldsymbol{w},\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega)$, $\forall q \in L^2_0(\Omega)$ and $\theta,\psi \in H^1(\Omega)$.

The next results show that the nonlinear forms O, $\mathcal{C}$ and D are continuous.

Proposition 2.3 *There exist positive constants C_{O}, C_{c} and C_{D} depending on Ω such that*

$$\begin{aligned} O(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) &\leq C_{O} \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} &(2.18)\\ \mathcal{C}(\boldsymbol{w},\theta,\phi) &\leq C_{c} \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \|\theta\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}, &(2.19)\\ D(\theta,\boldsymbol{v}) &\leq C_{D} \|\theta\|_{H^1(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, &(2.20)\end{aligned}$$

for all $\boldsymbol{w},\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega)$ and $\theta,\psi \in H^1(\Omega)$.

Proof. Applying Hölder's inequality, we have

$$|O(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})| \leq \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)}.$$

Estimate (2.18) follows from the compact imbedding (2.3) with $q = 4$. The proof of the continuity of $\mathcal{C}$ and D uses essentially the same techniques as for the form O, so we skip the details. □

2.4 A priori estimates and solvability

In this subsection, we prove the existence and uniqueness of solutions for the (2.4). We will apply the Leray-Schauder theorem, see [14, Theorem 11.3, p. 280]. For this, we must prove that $\mathscr{S}$ is well-defined, a compact operator on $\boldsymbol{K} \times H^1_0(\Omega)$ and that the set

$$\mathcal{Z} = \{(\boldsymbol{u},\theta) \in \boldsymbol{K} \times H^1_0(\Omega) : (\boldsymbol{u},\theta) = \lambda \mathscr{S}(\boldsymbol{u},\theta) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. For the convenience of the subsequent analysis, setting the following product norm on the space $\boldsymbol{K} \times H^1_0(\Omega)$:

$$\|(\boldsymbol{u},\theta)\| = \big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)} \big)^{1/2}.$$

The main result of this subsection is stated as follows.

Theorem 2.1 *Let $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$, $f \in L^2(\Omega)$. Then the problem (2.4) has at least one weak solution $(\boldsymbol{u},p,\theta) \in \boldsymbol{H}^1_T(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega)$ satisfying*

$$\|(\boldsymbol{u},\theta)\| \leq \kappa^{-1} \alpha_{M}^{-1} C_{F} \big(\|f\|_{L^2(\Omega)} + \nu^{-1} \alpha_{A}^{-1} C_{D} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \big). \tag{2.21}$$

Moreover, if $\min(\nu\alpha_{A}, \kappa\alpha_{M})^{-1} C_{D} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \leq 1$, the solution is unique.

Since solving (2.4) is equivalent to finding a fixed point of the map $\mathscr{S}$ defined in (2.8), we start by checking that the map $\mathscr{S}$ is well-defined, i.e. that there exists a unique solution $(\boldsymbol{u},\theta)$ to the linearized problem (2.9).

Lemma 2.2 *There exists a unique* $(\boldsymbol{u},\theta) \in \boldsymbol{H}^1_T(\Omega) \times H^1_0(\Omega)$ *satusfying the linearized problem* (2.9). *In addition, we have the following bound*

$$\|(\boldsymbol{u},\theta)\| \le \alpha_{\mathcal{A}}^{-1}\big(C_D\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\,\|(\boldsymbol{w},\phi)\| + C_F\,\|f\|_{L^2(\Omega)}\big) \tag{2.22}$$

Proof. First, we write the mixed system (2.9) in the following equivalent compact form:

Find $(\boldsymbol{u},\theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ such that:

$$\mathcal{A}((\boldsymbol{w},\phi),(\boldsymbol{u},\theta),(\boldsymbol{v},\psi)) \quad = \quad \mathcal{L}((\boldsymbol{w},\phi),(\boldsymbol{v},\psi)), \quad \forall(\boldsymbol{v},\psi) \in \boldsymbol{K} \times H^1_0(\Omega), \tag{2.23}$$

where the forms $\mathcal{A}$ and $\mathcal{L}$ are defined as

$$\mathcal{A}((\boldsymbol{w},\phi),(\boldsymbol{u},\theta),(\boldsymbol{v},\psi)) = A(\boldsymbol{u},\boldsymbol{v}) + M(\theta,\psi) + O(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) + \mathcal{C}(\boldsymbol{w},\theta,\psi), \tag{2.24}$$

and

$$\mathcal{L}((\boldsymbol{w},\phi),(\boldsymbol{v},\psi)) = D(\phi,\boldsymbol{v}) + F(\psi). \tag{2.25}$$

It follows from Proposition 2.1 that

$$\mathcal{A}((\boldsymbol{w},\phi),(\boldsymbol{u},\theta),(\boldsymbol{u},\theta)) \ge \nu\alpha_A\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \kappa\alpha_M\,\|\theta\|^2_{H^1(\Omega)} \ge \alpha_{\mathcal{A}}\,\|(\boldsymbol{u},\theta)\|^2\,, \tag{2.26}$$

where

$$\alpha_{\mathcal{A}} = \min(\nu\alpha_A, \kappa\alpha_M). \tag{2.27}$$

Moreover, employing the continuity results in (2.14)-(2.15), (2.18)-(2.19) together with the inequality $(a+b) \le 2^{1/2}(a^2+b^2)^{1/2}$, we can check that the trilinear form $\mathcal{A}$ verifies the estimate:

$$\begin{aligned}
&\mathcal{A}((\boldsymbol{w},\phi),(\boldsymbol{u},\theta),(\boldsymbol{v},\psi)) \le |A(\boldsymbol{u},\boldsymbol{v})| + |M(\theta,\psi)| + |O(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})| + |\mathcal{C}(\boldsymbol{w},\theta,\psi)|,\\
&\le \big(\nu C_A + \kappa C_M + (C_O + C_c)\,\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}\big)\big(\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\big)^{1/2}\big(\,\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\psi\|_{H^1(\Omega)}\big)^{1/2}\\
&\le C_{\mathcal{A}}\,\|(\boldsymbol{u},\theta)\|\,\|(\boldsymbol{v},\psi)\|\,,
\end{aligned}$$

with $C_{\mathcal{A}} = \big(\nu C_A + \kappa C_M + (C_O + C_c)\,\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}\big)$. Thus, we showed that $\mathcal{A}$ is coercive and continuous in $\boldsymbol{K} \times H^1_0(\Omega)$. In addition, thanks to (2.20) and (2.17), we have

$$\mathcal{L}((\boldsymbol{w},\phi),(\boldsymbol{v},\psi)) \le C_D\,\|\phi\|_{H^1(\Omega)}\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} + C_F\,\|f\|_{L^2(\Omega)}\,\|\psi\|_{H^1(\Omega)} \le C_{\mathcal{L}}\,\|(\boldsymbol{v},\psi)\|\,, \tag{2.28}$$

with $C_{\mathcal{L}} = C_D\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\,\|(\boldsymbol{w},\phi)\| + C_F\,\|f\|_{L^2(\Omega)}$. So, the form $\mathcal{L}$ is an element of $(\boldsymbol{K} \times H^1_0(\Omega))'$. Therefore, by Lax-Milgram's theorem, there exists a unique $(\boldsymbol{u},\theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ solution of (2.23) and equivalently to (2.9). The estimate (2.22) follows by combining the estimates in (2.26) and (2.28). □

Proof of Theorem 2.1. In order to prove the existence of the velocity $\boldsymbol{u}$ and the temperature θ, we proceed in two steps.

Step 1 : Let us show that the operator $\mathscr{S}$ is compact. Suppose $(\boldsymbol{w}_n,\phi_n) \in \boldsymbol{K} \times H^1_0(\Omega)$, $n \in \mathbb{N}$ such that $(\boldsymbol{w}_n,\phi_n) \rightharpoonup (\boldsymbol{w},\phi)$ weakly in $\boldsymbol{K} \times H^1_0(\Omega)$. Define $(\boldsymbol{u}_n,\theta_n) = \mathscr{S}(\boldsymbol{w}_n,\phi_n)$ and $(\boldsymbol{u},\theta) = \mathscr{S}(\boldsymbol{w},\phi)$. We will show that $(\boldsymbol{u}_n,\theta_n) \to (\boldsymbol{u},\theta)$ strongly in $\boldsymbol{K} \times H^1_0(\Omega)$.

A direct application of the coercivity of $\mathcal{A}$ in (2.26), (2.14) and (2.20) give

$$\begin{aligned}
\|(\boldsymbol{u}_n,\theta_n) - (\boldsymbol{u},\theta)\|^2 &\le \alpha_{\mathcal{A}}^{-1}\,\mathcal{A}((\boldsymbol{w}_n-\boldsymbol{w},\phi_n-\phi),(\boldsymbol{u}_n-\boldsymbol{u},\theta_n-\theta),(\boldsymbol{u}_n-\boldsymbol{u},\theta_n-\theta))\\
&\le \alpha_{\mathcal{A}}^{-1} D(\phi_n-\phi,\boldsymbol{u}_n-\boldsymbol{u})\\
&\le \alpha_{\mathcal{A}}^{-1}\Big(\,\|\phi_n-\phi\|_{L^4(\Omega)}\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\Big)
\end{aligned} \tag{2.29}$$

Since $\phi_n \rightharpoonup \phi$ weakly in $H^1(\Omega)$, therefore $\phi_n \to \phi$ strongly in $L^s(\Omega)$, for $1 \leq s < 6$. Moreover, we note that the sequence $(\boldsymbol{w}_n, \phi_n)_n$ is bounded. Since $(\boldsymbol{u}_n, \theta_n)$ and $(\boldsymbol{u}, \theta)$ satisfy (2.22), they are also bounded in the corresponding H^1 norm and then by continuity in the L^4 norm. We conclude from (2.29) that $(\boldsymbol{u}_n, \theta_n) \to (\boldsymbol{u}, \theta)$ strongly in $\boldsymbol{K} \times H^1(\Omega)$. Hence, we obtain compactness of the operator $\mathscr{S}$.

Step 2 : We show that there exists a constant $C_* > 0$ such that $\|(\boldsymbol{u}, \theta)\| \leq C_*$ for any $(\boldsymbol{u}, \theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ and for any $\lambda \in [0,1]$ such that $(\boldsymbol{u}, \theta) = \lambda \mathscr{S}(\boldsymbol{u}, \theta)$.

By the definition of $\mathscr{S}$, we have:

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) &= D(\lambda\theta, \boldsymbol{v}) - O(\lambda \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \\ M(\theta, \psi) &= F(\psi) - \mathcal{C}(\lambda \boldsymbol{u}, \theta, \psi), \quad \forall \psi \in H^1_0(\Omega). \end{aligned} \tag{2.30}$$

Choosing $(\boldsymbol{v}, \psi) = (\boldsymbol{u}, \theta)$ and using the properties (2.12) and (2.13), we obtain

$$A(\boldsymbol{u}, \boldsymbol{u}) = D(\lambda\theta, \boldsymbol{u}) \qquad \text{and} \qquad M(\theta, \theta) = F(\theta) \tag{2.31}$$

From the coercivity of A and M (c.f. (2.10) and (2.11)) and the continuity of D and F (c.f. (2.20) and (2.17)), we have immediately that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \lambda \nu^{-1} \alpha_A^{-1} C_D \|\theta\|_{H^1(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \quad \text{and} \quad \|\theta\|_{H^1(\Omega)} \leq \kappa^{-1} \alpha_M^{-1} C_F \|f\|_{L^2(\Omega)}.$$

Then, using the fact that $\lambda \leq 1$, it follows

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq K_1(\nu, \kappa, \boldsymbol{g}, f) \quad \text{and} \quad \|\theta\|_{H^1(\Omega)} \leq K_2(\kappa, f), \tag{2.32}$$

where $K_1(\nu, \kappa, \boldsymbol{g}, f)$ and $K_2(\kappa, f)$ are the following positive constants independent of $(\boldsymbol{u}, \theta)$ and λ:

$$K_1(\nu, \kappa, \boldsymbol{g}, f) = \nu^{-1} \alpha_A^{-1} C_D \|\theta\|_{H^1(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \quad \text{and} \quad K_2(\kappa, f) = \kappa^{-1} \alpha_M^{-1} C_F \|f\|_{L^2(\Omega)}. \tag{2.33}$$

Then, by the Leray-Schauder fixed point theorem, there exists a fixed point $(\boldsymbol{u}, \theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ for the operator $\mathscr{S}$ wich is a solution of (2.6). The estimate (2.21) follows directly from (2.32) and (2.33).

Now, we show the uniqueness of the solution of (2.4). Suppose that $(\boldsymbol{u}_1, \theta_1, p_1)$ and $(\boldsymbol{u}_2, \theta_2, p_2)$ are two solutions of (2.4). Then $(\boldsymbol{u}_1, \theta_1)$ and $(\boldsymbol{u}_2, \theta_2)$ are solutions of (2.6) or equivalently a fixed point of the operator $\mathscr{S}$. Denoting by $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\theta = \theta_1 - \theta_2$, applying the coercivity of the form $\mathcal{A}$ and the continuity of the form D as in (2.29), we obtain:

$$\begin{aligned} \|(\boldsymbol{u}, \theta)\|^2 &\leq \alpha_{\mathcal{A}}^{-1} \mathcal{A}((\boldsymbol{u}, \theta), (\boldsymbol{u}, \theta), (\boldsymbol{u}, \theta)) = \alpha_{\mathcal{A}}^{-1} D(\theta, \boldsymbol{u}) \\ &\leq \alpha_{\mathcal{A}}^{-1} C_D \left(\|\theta\|_{H^1(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \right) \\ &\leq \frac{1}{2} \alpha_{\mathcal{A}}^{-1} C_D \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \left(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)} \right) \leq \alpha_{\mathcal{A}}^{-1} C_D \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|(\boldsymbol{u}, \theta)\|^2, \end{aligned}$$

where $\alpha_{\mathcal{A}}$ id defined in (2.27). This implies that if $\alpha_{\mathcal{A}}^{-1} C_D \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} < 1$, we have $\boldsymbol{u} = \boldsymbol{0}$ and $\theta = 0$. Moreover, from (2.4a), we have for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$B(\boldsymbol{v}, p_1 - p_2) = D(\theta, \boldsymbol{v} - A(\boldsymbol{u}, \boldsymbol{v}) \underbrace{- O(\boldsymbol{u}_1; \boldsymbol{u}_1, \boldsymbol{v}) + O(\boldsymbol{u}_2; \boldsymbol{u}_2, \boldsymbol{v})}_{=0} = 0.$$

We then conclude by using the inf-sup condition in (2.7) that $p_1 - p_2 = 0$. This completes the proof.

3 The DG finite element approximation

In this section, we introduce a mixed DG approximation for the Boussinesq system (1.2). We provide the solvability and stability of the discrete scheme. To this end, let us introduce some notations. We assume The domain Ω is discretized by a discrete family of conforming meshes $\mathcal{T}_h$ made of tetrahedra. The index h is indicative of the mesh size h which is defined as $h = \max_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of T. The family is supposed to be regular in Ciarlet's sense [7], i.e. there exists $\varsigma > 0$ independent of h such that the ratio

$$\frac{h_T}{\rho_T} \leq \varsigma, \quad \forall T \in \mathcal{T}_h, \tag{3.1}$$

where ρ_T is the diameter of the inscribed circle in T. We shall use the assumption (3.1) throughout this work. Let us denote by $\mathcal{F}_h^I$ the set of internal faces and by $\mathcal{F}_h^\Gamma$ the set of external faces on Γ. We set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^\Gamma$. We denote by h_e the diameter of each face e. Let T^+ and T^- be two adjacent elements of $\mathcal{T}_h$ and let $\boldsymbol{n}^+$ (respectively $\boldsymbol{n}^-$) be the outward unit normal vector on ∂T^+ (respectively ∂T^-). For a vector field $\boldsymbol{u}$, we denote by $\boldsymbol{u}^\pm$ the trace of $\boldsymbol{u}$ from the interior of $T^\pm$. We define jumps

$$[\![\boldsymbol{v}]\!]_T := \boldsymbol{n}^+\times\boldsymbol{v}^+ + \boldsymbol{n}^-\times\boldsymbol{v}^-, \qquad [\![\boldsymbol{v}]\!]_N := \boldsymbol{v}^+\cdot\boldsymbol{n}^+ + \boldsymbol{v}^-\cdot\boldsymbol{n}^-, \quad [\![q]\!] := q^+\boldsymbol{n}^+ + q^-\boldsymbol{n}^-, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}^+ - \boldsymbol{v}^-,$$

and averages

$$\{\!\{\boldsymbol{v}\}\!\} := \frac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-), \qquad \{\!\{q\}\!\} := \frac{1}{2}(q^+ + q^-),$$

and adopt the convention that for boundary faces $e \in \mathcal{F}_h^\Gamma$, we set $[\![\boldsymbol{v}]\!]_T = \boldsymbol{v}\times\boldsymbol{n}$, $[\![\boldsymbol{v}]\!]_N = \boldsymbol{v}\cdot\boldsymbol{n}$, $[\![q]\!] = q\boldsymbol{n}$, $\{\!\{\boldsymbol{v}\}\!\} = \boldsymbol{v}$ and $\{\!\{q\}\!\} = q$. Let $\mathcal{P}_k$ denotes the space of polynomials of total degree at most k on T with $k = 1, 2$ or 3. The corresponding vector-valued function space is denoted by $\boldsymbol{\mathcal{P}}_k$.

3.1 Defining discrete problem

Now, we introduce the following finite element spaces which respectively approximate $\boldsymbol{u}$, $\boldsymbol{b}$ and p:

$$\begin{aligned}
\boldsymbol{X}_h &:= \left\{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\, \boldsymbol{v}_{h|T} \in \boldsymbol{\mathcal{P}}_k(T)\,,\, \forall T \in \mathcal{T}_h\right\}\\
Z_h &:= \left\{\psi_h \in L^2(\Omega);\, \psi_{h|T} \in \mathcal{P}_k(T)\,,\, \forall T \in \mathcal{T}_h\right\}\\
Q_h &:= \left\{q_h \in L_0^2(\Omega);\, q_{h|T} \in \mathcal{P}_{k-1}(T)\,,\, \forall T \in \mathcal{T}_h\right\}
\end{aligned}$$

We denote by W_h the product space $\boldsymbol{X}_h \times Z_h$. The norm $\|\cdot\|_{\boldsymbol{L}^2(\mathcal{T}_h)}$ is defined by

$$\|\cdot\|_{\boldsymbol{L}^2(\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} \|\cdot\|_{0,T}\,, \text{ for any } T \in \mathcal{T}_h,$$

with $\|\cdot\|_{0,T} = \|\cdot\|_{\boldsymbol{L}^2(T)}$. Similarly, we use the notation $\|\cdot\|_{0,e} = \|\cdot\|_{\boldsymbol{L}^2(e)}$ for any $e \in \mathcal{F}_h$.

The mixed DG scheme reads: Find $((\boldsymbol{u}_h, \theta_h), p_h) \in \boldsymbol{W}_h \times Q_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + O_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + B_h(\boldsymbol{v}_h, p_h) = D_h(\theta_h, \boldsymbol{v}_h), \tag{3.2a}$$

$$B_h(\boldsymbol{u}_h, q_h) = 0, \tag{3.2b}$$

$$M_h(\theta_h, \psi_h) + \mathcal{C}_h(\boldsymbol{u}_h, \theta_h, \psi_h) = F_h(\psi_h), \tag{3.2c}$$

for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$.

The bilinear form A_h is defined by

$$
\begin{aligned}
A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) \quad := \quad & \nu \sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\,\boldsymbol{u}_h\cdot\mathbf{curl}\,\boldsymbol{v}_h\,d\boldsymbol{x} + \nu\sum_{T\in\mathcal{T}_h}\int_T(\mathrm{div}\,\boldsymbol{u}_h)(\mathrm{div}\,\boldsymbol{v}_h)\,d\boldsymbol{x}\\
- \quad & \nu\sum_{e\in\mathcal{F}_h^I}\Big(\int_e\{\!\{\mathbf{curl}\,\boldsymbol{u}_h\}\!\}\cdot[\![\boldsymbol{v}_h]\!]_T\,ds + \int_e\{\!\{\mathbf{curl}\,\boldsymbol{v}_h\}\!\}\cdot[\![\boldsymbol{u}_h]\!]_T\,ds\Big)\\
- \quad & \nu\sum_{e\in\mathcal{F}_h}\Big(\int_e\{\!\{\mathrm{div}\,\boldsymbol{u}_h\}\!\}[\![\boldsymbol{v}_h]\!]_N\,ds + \int_e\{\!\{\mathrm{div}\,\boldsymbol{v}_h\}\!\}[\![\boldsymbol{u}_h]\!]_N\,ds\Big)\\
+ \quad & \nu\sum_{e\in\mathcal{F}_h^I}\frac{\sigma}{h_e}\int_e[\![\boldsymbol{u}_h]\!]_T\cdot[\![\boldsymbol{v}_h]\!]_T\,ds + \nu\sum_{e\in\mathcal{F}_h}\frac{\sigma}{h_e}\int_e[\![\boldsymbol{u}_h]\!]_N[\![\boldsymbol{v}_h]\!]_N\,ds
\end{aligned}
$$

with $\sigma_1 > 0$ a stabilization parameter that will be chosen large enough. The two last terms in the definition of A_h involving the tangential and normal jumps of the discrete vector fields across the edges are necessary to ensure the coercivity of the bilinear form A_h.

We define the convective term in the Navier-Stokes equations with :

$$
O_h(\boldsymbol{w}_h,\boldsymbol{u}_h,\,\boldsymbol{v}_h) := \sum_{T\in\mathcal{T}_h}\int_T(\mathbf{curl}\,\boldsymbol{w}_h\times\boldsymbol{u}_h)\cdot\boldsymbol{v}_h\,d\boldsymbol{x} \tag{3.3}
$$

The divergence constraint on the velocity is represented by B_h :

$$
B_h(\boldsymbol{v}_h,q_h) := -\sum_{T\in\mathcal{T}_h}\int_T(\mathrm{div}\,\boldsymbol{v}_h)q_h\,d\boldsymbol{x} + \sum_{e\in\mathcal{F}_h}\int_e\{\!\{q_h\}\!\}[\![\boldsymbol{v}_h]\!]_N\,ds. \tag{3.4}
$$

The coupling form D_h is defined by

$$
D_h(\theta_h,\boldsymbol{v}_h) = \sum_{T\in\mathcal{T}_h}\int_T(\theta_h\boldsymbol{g})\cdot\boldsymbol{v}_h
$$

For the heat equation, we define the form M_h by:

$$
\begin{aligned}
M_h(\theta_h,\psi_h) \quad := \quad & \kappa\sum_{T\in\mathcal{T}_h}\int_T\nabla\theta_h\cdot\nabla\psi_h\,d\boldsymbol{x} - \kappa\sum_{e\in\mathcal{F}_h}\Big(\int_e\{\!\{\nabla\theta_h\}\!\}\cdot[\![\psi_h]\!] + \sum_{e\in\mathcal{F}_h}\int_e\{\!\{\nabla\psi_h\}\!\}\cdot[\![\theta_h]\!]\Big)\\
+ \quad & \sum_{e\in\mathcal{F}_h}\frac{\kappa\gamma}{h_e}\int_e[\![\theta_h]\!]\cdot[\![\psi_h]\!]\,ds
\end{aligned}
$$

with $\gamma > 0$ is a stabilization parameter that will be chosen large enough to ensure the coercivity of the bilinear form M_h.

We use the upwinding of Lesaint-Raviart [17] to discretize the convection term $\boldsymbol{u}\cdot\nabla\theta$. So, the coupling form C_h using upwind fluxes is defined by :

$$
\begin{aligned}
C_h(\boldsymbol{w}_h,\theta_h,\psi_h) :=& \sum_{T\in\mathcal{T}_h}\Big(\int_T(\boldsymbol{w}_h\cdot\nabla\theta_h)\psi_h\,d\boldsymbol{x} + \int_{\partial T_-}|\,\{\!\{\boldsymbol{w}_h\}\!\}\cdot\boldsymbol{n}_T|(\theta_h^{int}-\theta_h^{ext})\psi_h^{int}\,d\boldsymbol{s}\Big),\\
&+\frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_T(\mathrm{div}\,\boldsymbol{w}_h)\theta_h\cdot\psi_h\,d\boldsymbol{x} - \frac{1}{2}\sum_{e\in\mathcal{F}_h}\int_e[\![\boldsymbol{w}_h]\!]_N\,\{\!\{\theta_h\cdot\psi_h\}\!\}\,d\boldsymbol{s},
\end{aligned} \tag{3.5}
$$

where θ_h^{int}, ψ_h^{int} (respectively θ_h^{ext}) refers to the trace of θ_h and ψ_h (respectively of θ_h) on a side of T

taken from the interior of T (respectively taken from the exterior of T on that side). ∂T_- denotesthe inflow boundary of T defined by

$$\partial T_- = \{\boldsymbol{x} \in \partial T : \ \{\!\{\boldsymbol{u}\}\!\} \cdot \boldsymbol{n} < 0\}.$$

Finally, the form F_h is defined by

$$F_h(\psi_h) = \int_\Omega f \psi_h \, d\boldsymbol{x}.$$

3.2 Preliminary results

We need to introduce the following semi-norms

$$|\boldsymbol{v}_h|^2_{1,h} := \sum_{T\in\mathcal{T}_h} \|\mathrm{div}\, \boldsymbol{v}_h\|^2_{0,T} + \sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|^2_{0,T}\,, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.$$

and

$$|||\boldsymbol{v}_h|||^2 := |\boldsymbol{v}_h|^2_{1,h} + \sum_{e\in\mathcal{F}_h} \frac{\sigma}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|^2_{0,e} + \sum_{e\in\mathcal{F}^I_h} \frac{\sigma}{h_e} \|[\![\boldsymbol{v}_h]\!]_T\|^2_{0,e}\,, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.$$

$$\|\psi_h\|^2_h := \sum_{T\in\mathcal{T}_h} \|\nabla\psi_h\|^2_{0,T} + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{h_e} \|[\![\psi_h]\!]\|^2_{0,e}\,, \qquad \forall \psi_h \in Z_h.$$

$$\|q_h\|^2_{Q_h} := \|q_h\|^2_{0,\Omega}\,, \qquad \forall q_h \in Q_h.$$

The proof of the following Lemma follows along the same lines as Lemma 10.2.1 in [28]. We carry out the details for the sake of completeness.

Lemma 3.1 *For σ and γ large enough, there exists positive constants α_{A_h}, α_{M_h} independent of h such that:*

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geqslant \alpha_{A_h} \nu |||\boldsymbol{u}_h|||^2, \qquad \forall \boldsymbol{u}_h \in \boldsymbol{X}_h, \tag{3.6a}$$

$$M_h(\theta_h, \theta_h) \geqslant \alpha_{M_h} \kappa \|\theta_h\|^2_h\,, \qquad \forall \theta_h \in Z_h. \tag{3.6b}$$

Proof. For the proof of (3.6a), we use similar arguments in [25]. Let $\boldsymbol{u}_h \in \boldsymbol{X}_h$, we have:

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) &= \nu \sum_{T\in\mathcal{T}_h} (\|\mathbf{curl}\, \boldsymbol{u}_h\|^2_{0,T} + \|\mathrm{div}\, \boldsymbol{u}_h\|^2_{0,T}) \\ &- 2\nu \sum_{e\in\mathcal{F}^I_h} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds - 2\nu \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\mathrm{div}\, \boldsymbol{u}_h\}\!\} [\![\boldsymbol{u}_h]\!]_N \, ds \\ &+ \nu \sum_{e\in\mathcal{F}^I_h} \frac{\sigma}{h_e} \|[\![\boldsymbol{u}_h]\!]_T\|^2_{0,e} + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma}{h_e} \|[\![\boldsymbol{u}_h]\!]_N\|^2_{0,e}\,. \end{aligned} \tag{3.7}$$

Using the Cauchy-Schwarz inequality, we have:

$$2\nu \sum_{e\in\mathcal{F}^I_h} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds \quad \leq \quad 2C_1\nu \Big(\sum_{T\in\mathcal{T}_h} \|\ \mathbf{curl}\ \boldsymbol{u}_h\|^2_{0,T}\Big)^{1/2} \Big(\sum_{e\in\mathcal{F}^I_h} h_e^{-1} \|[\![\boldsymbol{u}_h]\!]_T\|^2_{0,e} \Big)^{\frac{1}{2}}$$

$$\leq \quad 2C_1\nu|||\boldsymbol{u}_h|||\Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1}\left\|[\![\boldsymbol{u}_h]\!]_T\right\|_{0,e}^2\Big)^{\frac{1}{2}}.$$

In the same way, we get

$$2\nu\sum_{e\in\mathcal{F}_h}\int_e \{\!\{\operatorname{div}\boldsymbol{u}_h\}\!\}[\![\boldsymbol{u}_h]\!]_N\,ds \leq 2C_2\nu|||\boldsymbol{u}_h|||\Big(\sum_{e\in\mathcal{F}_h} h_e^{-1}\left\|[\![\boldsymbol{u}_h]\!]_N\right\|_{0,e}^2\Big)^{\frac{1}{2}},$$

where C_1 and C_2 are a positive constants independent of h.

Applying now Young's inequality, we obtain

$$A_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \quad\geq\quad \nu\Big(1-\frac{C}{\sigma}\Big)|||\boldsymbol{u}_h|||^2,$$

with $C = C_1 + C_2$. Therefore, the estimate (3.6a) follows by assuming that $\sigma > C$.

For the proof of (3.6b), we refer the reader to [8, 17]

□

We will need the following inf-sup condition on B_h, where the proof can be found in [25, 28].

Lemma 3.2 *There exists $\beta > 0$ only depending on Ω such that :*

$$\inf_{q_h\in Q_h}\sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\frac{B_h(\boldsymbol{v}_h,q_h)}{|||\boldsymbol{v}_h|||\,\|q_h\|_{Q_h}} \geqslant \beta. \tag{3.8}$$

Lemma 3.3 *Let $\boldsymbol{u}_h$, $\boldsymbol{v}_h \in X_h$, θ_h, $\psi_h \in Z_h$ and $p_h \in Q_h$. Then, we have :*

$$|A_h(\boldsymbol{u}_h,\boldsymbol{v}_h)| \leqslant C_{A_h}\,\nu\,|||\boldsymbol{u}_h|||\;|||\boldsymbol{v}_h|||, \tag{3.9a}$$
$$|M_h(\theta_h,\psi_h)| \leqslant C_{M_h}\kappa\;\|\theta_h\|_h\;\;\|\psi_h\|_h\,, \tag{3.9b}$$
$$|B_h(\boldsymbol{v}_h,p_h)| \leqslant C_{B_h}|||\boldsymbol{v}_h|||\;\|p_h\|_{0,\Omega}\,, \tag{3.9c}$$
$$|F_h(\psi_h)| \leqslant C_{F_h}\;\|f\|_{L^2(\Omega)}\;\|\psi_h\|_h\,, \tag{3.9d}$$

where $C_{A_h}, C_{M_h}, C_{B_h}$ and C_{F_h} are positive constants independent of h.

Proof. The proof of the continuity properties (3.9b), (3.9c) can b found in [8, 17, 23] and the continuity of A_h and F_h follows from the Cauchy-Schwarz inequalityhas been proved in [25]. □

Next, we recall the following discrete Sobolev inequality for discontinuous finite element spaces (see [17, Lemma 3.1]) :

$$\forall\psi_h \in Z_h, \quad \|\psi_h\|_{\boldsymbol{L}^6(\Omega)} \leq C\,\|\psi_h\|_h\,, \tag{3.10}$$

where the constant $C > 0$ is independent of h. This inequality is not still valid when the norm $\|\cdot\|_h$ is replaced by $|||\cdot|||$ without supposing more regularity on the boundary Γ. When it is Lipschitz polyhedral, we have the following discrete Poincaré-Friedrichs inequality (see [30, Lemma 3.1])

$$\forall\boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C|||\boldsymbol{v}_h|||. \tag{3.11}$$

When the boundary Γ is of class $\mathcal{C}^{2,1}$, an extention of the previous inequality is proved in [26, Lemma 3.4.2] for the $\boldsymbol{L}^p$ normn with $p \in (1, 6]$:

$$\forall\boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^p(\Omega)} \leq C|||\boldsymbol{v}_h|||. \tag{3.12}$$

The discret Sobolev inequlity (3.12) plays an important role in the stability of our DG sheme. Indeed, with $p = 4$, this ineqlity allows to apply an $L^4 - L^2 - L^4$ to bound the terms on T in the nonlinear formes O_h and $\mathcal{C}_h$. So, starting now, we shall make the additional smoothness assumptions on Ω. Indeed, we suppose that the domain Ω has a boundary of class $\mathcal{C}^{2,1}$.

Lemma 3.4 *We have*

$$\forall \theta_h \in Z_h, \forall \boldsymbol{u}_h \in \boldsymbol{X}_h \quad |D_h(\theta_h, \boldsymbol{u}_h)| \leqslant C_{D_h} \; \|\boldsymbol{g}\|_{0,\Omega} \, \|\theta_h\|_h \; |||\boldsymbol{v}_h|||, \tag{3.13a}$$

$$\forall \boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad |O_h(\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)| \leqslant C_{O_h} |||\boldsymbol{w}_h||| \, |||\boldsymbol{u}_h||| \, |||\boldsymbol{v}_h||| \tag{3.13b}$$

$$\forall \boldsymbol{w}_h \in \boldsymbol{X}_h, \forall \theta_h, \psi_h \in Z_h, \quad |C_h(\boldsymbol{w}_h, \theta_h, \psi_h)| \leqslant C_{c_h} |||\boldsymbol{w}_h||| \; \|\theta_h\|_h \; \|\psi_h\|_h \tag{3.13c}$$

where C_{D_h}, C_{O_h} and C_{c_h} are positive constants independent of h.

Proof. Using Hölder's inequality, we can easily obtain

$$|D_h(\theta_h, \boldsymbol{v}_h)| \leq \|g\|_{0,\Omega} \, \|\theta_h\|_{L^6(\Omega)} \, \|\boldsymbol{v}_h\|_{L^3(\Omega)} \tag{3.14}$$

Thanks to (3.10) and to (3.12) with $p = 3$, we have (3.13a).

Applying Cauchy-Schwarz inequality, we obtain

$$|O_h(\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)| = \Big| \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w}_h \times \boldsymbol{u}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} \Big| \quad \leq \quad 2 \, \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^2(\mathcal{T}_h)} \, \|\boldsymbol{u}_h\|_{\boldsymbol{L}^3(\Omega)} \, \|\boldsymbol{v}_h\|_{\boldsymbol{L}^6(\Omega)} \, .$$

By virtue of (3.12), the estimate (3.13b) follows immediately.

Now, applying the Sobolev embedding (3.12) with $p = 4$ and the trace inequalities (see for instance [8, Proposition 4.2]), we get (3.13c). □

We define the space

$$\boldsymbol{K}_h = \{\boldsymbol{v}_h \in \boldsymbol{X}_h; \quad B_h(\boldsymbol{u}_h, q_h) = 0, \quad \forall q_h \in Q_h\}$$

and the discrete energy norm on $\boldsymbol{K}_h \times Z_h$:

$$\|(\boldsymbol{u}_h, \theta_h)\|_h = \left(|||\boldsymbol{u}_h|||^2 + \|\theta_h\|_h^2 \right)^{1/2}$$

It is well-known (see, e.g., [17], [18]) that

$$C_h(\boldsymbol{w}_h, \theta_h, \theta_h) := \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T_-} | \{\!\{\boldsymbol{w}_h\}\!\} \cdot \boldsymbol{n}_T | \|\theta_h^{int} - \theta_h^{ext}\|^2 \, d\boldsymbol{s} + \frac{1}{2} \int_{\Gamma_+} |\boldsymbol{w}_h \cdot \boldsymbol{n}_T| \|\theta_h\|^2 \, d\boldsymbol{s} \geq 0. \tag{3.15}$$

Moreover, since the vector $\mathbf{curl}\, \boldsymbol{w}_h \times \boldsymbol{v}_h$ is orthogonal to $\boldsymbol{v}_h$, it is clear that for any $\boldsymbol{w}_h$, $\boldsymbol{u}_h$, $\boldsymbol{v}_h$ in $\boldsymbol{X}_h$

$$O_h(\boldsymbol{w}_h, \boldsymbol{v}_h, \boldsymbol{v}_h) = \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w}_h \times \boldsymbol{v}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} = 0, \tag{3.16}$$

To establish the well posedness of the DG scheme, we first eliminate the pressure from the problem by restricting ourselves to the space $\boldsymbol{K}_h$. So, we consider the following problem: Find $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + O_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = D_h(\theta_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h, \tag{3.17a}$$

$$M_h(\theta_h, \psi_h) + \mathcal{C}_h(\boldsymbol{u}_h, \theta_h, \psi_h) = F_h(\psi_h), \quad \forall \psi_h \in Z_h. \tag{3.17b}$$

We define the operator $\mathscr{S}_h$ by:

$$\mathscr{S}_h \colon \boldsymbol{K}_h \times Z_h \to \boldsymbol{K}_h \times Z_h$$
$$(\boldsymbol{w}_h, \phi_h) \mapsto \mathscr{S}_h(\boldsymbol{w}_h, \phi_h) = (\boldsymbol{u}_h, \theta_h),$$

where $(\boldsymbol{u}_h, \theta_h)$ is the solution of the linearized problem: Find $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + O_h(\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = \qquad D_h(\phi_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h, \tag{3.18a}$$

$$M_h(\theta_h, \psi_h) + \mathcal{C}_h(\boldsymbol{w}_h, \theta_h, \psi_h) = \qquad F_h(\psi_h), \quad \forall \psi_h \in Z_h. \tag{3.18b}$$

Then, the problem (3.17a)-(3.17b) can be rewritten equivalently as the following fixed-point problem:

$$\text{Find } (\boldsymbol{u}_h,\, \theta_h) \in \boldsymbol{K}_h \times Z_h \quad \text{such that} \quad \mathscr{S}_h(\boldsymbol{w}_h, \phi_h) = (\boldsymbol{u}_h,\, \theta_h). \tag{3.19}$$

3.3 Solvability of the fixed-point problem

In this subsection, we proceed analogously to Subsection 2.4 and prove the well-posedness of problem (3.17) by means of a fixed point argument. We will apply Brower's fixed point theorem (c.f. [6, Theorem 9.9-2]) instead of Shauder's theorem for the existence and the uniqueness of $(\boldsymbol{u}_h,\, \theta_h)$. The existence and uniqueness of p_h follow from the discrete inf-sup condition for the incompressibility form B_h. The classical Brouwer's fixed point theorem is stated as follows: let W be a nonempty compact convex subset of a finite-dimensional normed space, and let $S : W \to W$ be a contraction from W into itself. Then S has a unique fixed point in W.

We begin by the following result, where we show that the operator $\mathscr{S}_h$ is defined correctly.

Lemma 3.5 *Assuming the stabilization parameters σ and γ sufficiently large, there exists a unique solution $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h$ for the linearized problem (3.18). Moreover, we have the following estimates:*

$$|||\boldsymbol{u}_h||| \le \alpha_{A_h}^{-1}\nu^{-1}C_{D_h}\,\|\phi_h\|_h\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \qquad \text{and} \qquad \|\theta_h\|_h \le \alpha_{M_h}^{-1}\kappa^{-1}C_{F_h}\,\|f\|_{L^2(\Omega)}\,. \tag{3.20}$$

Proof. Since (3.18a)-(3.18b) consists of a linear system, it suffices to establish the uniqueness. Assuming that the data are homogeneous $f = 0$ and $\boldsymbol{g} = \boldsymbol{0}$, we can choose $(\boldsymbol{v}_h, \psi_h) = (\boldsymbol{u}_h, \theta_h)$ as test functions. This leads to

$$\alpha_{A_h}\nu|||\boldsymbol{u}_h|||^2 + \alpha_{M_h}\kappa\,\|\theta_h\|_h^2 = 0,$$

implying that $\boldsymbol{u}_h = \boldsymbol{0}$ and $\theta = 0$. The estimates in (3.20) follow as a consequence of the coercivity of A_h and M_h in (3.6) together with the continuity of D_h and F_h in (3.13a) and (3.9d). □

Theorem 3.1 *Assume that*

$$M := \max\left(\sqrt{2}\Big(\frac{C_{O_h}C_1(\nu,\kappa,f,\boldsymbol{g})}{\alpha_{A_h}\nu} + \frac{C_{\mathcal{C}_h}C_2(\kappa,f)}{\alpha_{M_h}\kappa}\Big), \sqrt{2}\frac{C_{D_h}}{\alpha_{A_h}\nu}\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\right) < 1 \tag{3.21}$$

Then, the problem (3.17) has a unique solution $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h$ satisfying

$$\nu|||\boldsymbol{u}_h|||^2 + \kappa\,\|\theta_h\|_h^2 \le \alpha_{M_h}^{-2}C_{F_h}^2\,\|f\|_{L^2(\Omega)}^2\left(\kappa^{-2}\alpha_{A_h}^{-2}C_{D_h}^2\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}^2 + 1\right) \tag{3.22}$$

Proof. The well-posedness of (3.17) follow by showing that operator $\mathscr{S}_h$ has a unique fixed-point. Setting $(\boldsymbol{v}_h, \psi_h) = (\boldsymbol{u}_h, \theta_h)$ in (3.17) and applying the coercivity of the forms A_h and M_h, the continuity of the form D_h and F_h, we obtain that

$$|||\boldsymbol{u}_h||| \le \alpha_{A_h}^{-1}\nu^{-1}C_{D_h}\,\|\theta_h\|_h\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \qquad \text{and} \qquad \|\theta_h\|_h \le \alpha_{M_h}^{-1}\kappa^{-1}C_{F_h}\,\|f\|_{L^2(\Omega)}$$

and then,

$$|||\boldsymbol{u}_h||| \leq C_1(\nu,\kappa,f,\boldsymbol{g}) \qquad \text{and} \qquad \|\theta_h\|_h \leq C_2(\kappa,f), \tag{3.23}$$

where

$$C_1(\nu,\kappa,f,\boldsymbol{g}) = \frac{C_{F_h} C_{D_h}}{\nu\kappa\alpha_{A_h}\alpha_{M_h}} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|f\|_{L^2(\Omega)} \qquad \text{and} \qquad C_2(\kappa,f) = \frac{C_{F_h}}{\kappa\alpha_{M_h}} \|f\|_{L^2(\Omega)}. \tag{3.24}$$

Inspired by this, define the compact and convex subset B_R of $\boldsymbol{K}_h \times Z_h$:

$$B_R = \Big\{(\boldsymbol{u}_h,\theta_h) \in \boldsymbol{K}_h \times Z_h \ : \ \|(\boldsymbol{u}_h,\theta_h)\|_h \leq R\Big\},$$

where $R = \alpha_{M_h}^{-1}\kappa^{-1}C_{F_h}\|f\|_{L^2(\Omega)}\big(\alpha_{A_h}^{-1}\nu^{-1}C_{D_h}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} + 1\big)$. It is easy to see that $\mathscr{S}_h$ maps B_R into B_R. Now, we show that the operator $\mathscr{S}_h$ is a contraction on B_R. To that end, let $(\boldsymbol{u}_h^1,\theta_h^1)$, $(\boldsymbol{u}_h^2,\theta_h^2) \in B_R$ and $(\boldsymbol{u}_h^1,\theta_h^1) = \mathscr{S}(\boldsymbol{w}_1,\phi_1)$, $(\boldsymbol{u}_h^2,\theta_h^2) = \mathscr{S}(\boldsymbol{w}_2,\phi_2)$. Setting $\boldsymbol{u}_h := \boldsymbol{u}_h^1 - \boldsymbol{u}_h^2$, $\theta_h := \theta_h^1 - \theta_h^2$, by definition, we deduce that

$$\begin{aligned}&A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + M_h(\theta_h,\phi_h) + O_h(\boldsymbol{w}_1,\boldsymbol{u}_h,\boldsymbol{v}_h) + \mathcal{C}_h(\boldsymbol{w}_1,\theta,\psi_h)\\ &= -O_h(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{u}_h^2,\boldsymbol{v}_h) - \mathcal{C}_h(\boldsymbol{w}_1-\boldsymbol{w}_2,\theta_h^2,\psi_h) + D_h(\phi_1-\phi_2,\boldsymbol{v}_h).\end{aligned}$$

Taking $(\boldsymbol{v}_h,\psi_h) = (\boldsymbol{u}_h,\theta_h)$, using (3.6) and (3.13), we obtain

$$\begin{aligned}|||\boldsymbol{u}_h|||^2 + \|\theta_h\|_h^2 \ &\leq\ \alpha_{A_h}^{-1}\nu^{-1}C_{O_h}|||\boldsymbol{w}_1-\boldsymbol{w}_2|||\ |||\boldsymbol{u}_h^2|||\ |||\boldsymbol{u}_h||| + \alpha_{M_h}^{-1}\kappa^{-1}C_{\mathcal{C}_h}|||\boldsymbol{w}_1-\boldsymbol{w}_2|||\,\big\|\theta_h^2\big\|_h\|\theta_h\|_h\\ &\leq\ \alpha_{A_h}^{-1}\nu^{-1}C_{D_h}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\phi_1-\phi_2\|_h\,|||\boldsymbol{u}_h|||\end{aligned}$$

Applying Young's inequality and using the fact that $(\boldsymbol{u}_h^2,\theta_h^2)$ satisfies (3.23), we derive that

$$\begin{aligned}|||\boldsymbol{u}_h|||^2 + \|\theta_h\|_h^2 \ &\leq\ 2\Big(\alpha_{A_h}^{-2}\nu^{-2}C_{O_h}^2C_1^2(\nu,\kappa,f,\boldsymbol{g}) + \alpha_{M_h}^{-2}\kappa^{-2}C_{\mathcal{C}_h}^2C_2^2(\kappa,f)\Big)|||\boldsymbol{w}_1-\boldsymbol{w}_2|||^2\\ &+\ 2\alpha_{A_h}^{-2}\nu^{-2}C_{D_h}^2\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}^2\|\phi_1-\phi_2\|_h^2,\end{aligned}$$

and then

$$\|(\boldsymbol{u}_h,\theta_h)\|_h \ \leq\ M\,\|(\boldsymbol{w}_1,\phi_1)-(\boldsymbol{w}_2,\phi_2)\|_h,$$

with

$$M = \max\Big(\sqrt{2}\big(\alpha_{A_h}^{-1}\nu^{-1}C_{O_h}C_1(\nu,\kappa,f,\boldsymbol{g}) + \alpha_{M_h}^{-1}\kappa^{-1}C_{\mathcal{C}_h}C_2(\kappa,f)\big), \sqrt{2}\alpha_{A_h}^{-1}\nu^{-1}C_{D_h}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\Big). \tag{3.25}$$

So, if $M < 1$ that is, if the smallness condition (3.21) is satisfied, the mapping $\mathscr{S}_h$ is a contraction. Consequently, an application of Brower's fixed point theorem shows that $\mathscr{S}_h$ has a unique fixed point in B_R, which is the solution of problem (3.17). Besides, the stability bound for $(\boldsymbol{u}_h,\theta_h)$ follows immediately. □

Remark 3.1 *We note that the hypothesis (3.21) is not necessary to get the existence of solution $(\boldsymbol{u}_h,\theta_h)$. Indeed, We can use only the fact that $\mathscr{S}_h$ is Lipschitz continuous on $\boldsymbol{K}_h \times Z_h$.*

Now, in order to recovering the pressure, we have the following Corollary.

Corollary 3.1 *Let $(\boldsymbol{u}_h,\theta_h)$ the solution of the problem (3.17) given by Theorem 3.1. Then, there existes a unique $p_h \in Q_h$ such that $((\boldsymbol{u}_h,\theta_h),p_h) \in \boldsymbol{X}_h \times Z_h \times Q_h$ is the solution of (3.2). Moreover, we have the following estimate:*

$$\|p_h\|_{0,\Omega} \leq C_3(\nu,\kappa,f,\boldsymbol{g}) \tag{3.26}$$

where

$$C_3(\nu,\kappa,f,\boldsymbol{g}) = \beta^{-1}C_{A_h}C_1(\nu,\kappa,f,\boldsymbol{g}) + \beta^{-1}C_{O_h}C_1^2(\nu,\kappa,f,\boldsymbol{g}) + \beta^{-1}C_{D_h}C_2(\kappa,f)\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \tag{3.27}$$

and $C_1(\nu,\kappa,f,\boldsymbol{g})$, $C_2(\kappa,f)$ are the constants in (3.23).

Proof. Analogously to the continuous case (see Lemma (2.1)), due to the inf-sup condition (3.8) and the continuity properties of A_h, O_h and D_h, here we also obtain that both (3.2) and (3.17) are equivalent. Indeed, the pressure is uniquely solvable by the following problem

$$B_h(\boldsymbol{v}_h, p_h) = -A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - O_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + D_h(\theta_h, \boldsymbol{v}_h). \tag{3.28}$$

Then, it suffices to prove the estimate (3.26) to conclude the proof. To do that, using the inf sup-condition (3.8) and ..., we obtain

$$\begin{aligned}
\beta \|p_h\|_{Q_h} &\leq \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{B_h(\boldsymbol{v}_h, p_h)}{|||\boldsymbol{v}_h|||} \\
&= \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \left\{ \frac{-A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - O_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + D_h(\theta_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||} \right\} \\
&\leq C_{A_h} |||\boldsymbol{u}_h||| + C_{O_h} |||\boldsymbol{u}_h|||^2 + C_{D_h} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|\theta_h\|_h
\end{aligned}$$

Recalling that $(\boldsymbol{u}_h, \theta_h) \in B_R$ and satisfies (3.23), then we obtain the estimate (3.26) from the latter estimate which concludes the proof. □

4 Error analysis

Let us begin by introducing an approximation result for the space $\boldsymbol{X}_h$ (see [15]). For $k = 1, 2, 3$, there exists a continuous interpolation operator I_h defined from $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{X}_h$ such that, for all $T \in \mathcal{T}_h$ and $e \in \mathcal{F}_h$:

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\ \ \forall q_h \in \mathcal{P}_{k-1}(T),\ \int_T q_h \operatorname{div}(I_h(\boldsymbol{v}) - \boldsymbol{v})\, dx = 0, \tag{4.1}$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega),\ \forall e \in \mathcal{F}_h,\ \ \forall q_h \in \mathcal{P}_{k-1}(e),\ \ \int_e q_h\, [\![I_h(\boldsymbol{v})]\!]\, ds = 0. \tag{4.2}$$

Moreover, for $s \in [1, k+1]$ the following interpolation estimate holds:

$$\forall \boldsymbol{v} \in \boldsymbol{H}^s(\Omega), \quad \|I_h(\boldsymbol{v}) - \boldsymbol{v}\|_{1,T} \leqslant C h_T^{s-1} \|\boldsymbol{v}\|_{s,\Delta_T}, \tag{4.3}$$

where Δ_T is a suitable macro-element containing T.

We use the L^2-projection of degree $k-1$ onto Q_h (respectively of degree k onto Z_h) to approximate the pressure p (respectively the temperature θ). So there exists approximationz $\Pi_Q \in Q_h$ and $\Pi_Z \in Z_h$ (see [17]), defined on each $T \in \mathcal{T}_h$ by

$$\forall q \in \mathcal{P}_{k-1}(T), \quad \int_T q(p - \Pi_Q p)\, d\boldsymbol{x} = 0. \tag{4.4a}$$

$$\forall \psi \in \mathcal{P}_k(T), \quad \int_T \psi(\theta - \Pi_Z \theta)\, d\boldsymbol{x} = 0. \tag{4.4b}$$

and satisy the following approximations properties for every integer $s \in [0, k]$:

$$\|p - \Pi_Q p\|_{0,T} \leq C h_T^s \|p\|_{s,T}, \qquad \forall p \in H^s(\Omega) \cup L^2_0(\Omega). \tag{4.5}$$

Theorem 4.1 *Let $(\boldsymbol{u}, \theta, p) \in \boldsymbol{H}^{k+1}(\Omega) \times H^{k+1}(\Omega) \times H^k(\Omega)$ be a solution of the continuous problem (1.2). Let $(\boldsymbol{u}_h, p_h, \boldsymbol{b}_h)$ be the solution in $\boldsymbol{X}_h \times Q_h \times \boldsymbol{C}_h$ of the DG problem (3.2). We assume that*

$$\frac{C_{O_h} K_1(\nu, \kappa, \boldsymbol{g}, f)}{\nu \alpha_{A_h}} + \frac{C_{D_h} \|\boldsymbol{g}\|_{0,\Omega}}{\nu \alpha_{A_h} \kappa \alpha_{M_h}} \leq \frac{1}{2}. \tag{4.6}$$

Then, there exists positive constant $C > 0$, independent of h, such that, the following estimates hold true:

$$\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|+\|\theta-\theta_h\|_h \quad \leqslant Ch^k\Big(\|\boldsymbol{u}\|_{k+1,\Omega}+\|\theta\|_{k+1,\Omega}+\|p\|_{k,\Omega}\Big) \tag{4.7}$$

$$\|p-p_h\|_{0,\Omega} \leq Ch^k\Big(\|\boldsymbol{u}\|_{k+1,\Omega}+\|\theta\|_{k+1,\Omega}+\|p\|_{k,\Omega}\Big) \tag{4.8}$$

Proof. We denote the corresponding errors by

$$\boldsymbol{e}_u := \boldsymbol{u}-\boldsymbol{u}_h, \quad e_\theta := \theta-\theta_h, \quad e_p := p - p_h.$$

We decompose these errors into

$$\boldsymbol{e}_u = \boldsymbol{\eta}_u + \boldsymbol{\chi}_u, \quad e_\theta = \eta_\theta + \chi_\theta, \quad e_p = \eta_p + \chi_p, \tag{4.9}$$

with

$$\begin{aligned} \boldsymbol{\eta}_u &:= \boldsymbol{u} - I_h\boldsymbol{u}, & \boldsymbol{\chi}_u &:= I_h\boldsymbol{u}-\boldsymbol{u}_h \\ \eta_\theta &:= \theta - \Pi_Z\theta, & \chi_\theta &:= \Pi_Z\theta - \theta_h \\ \eta_p &:= p - \Pi_Q p, & \chi_p &:= \Pi_Z p - p_h \end{aligned}$$

From the approximation properties of operators I_h, Π_Q and Π_Z (see (4.3), (4.4a) and (4.4b) respectively) combined with inverse inequality, it can be readily seen that

$$\|\!|\boldsymbol{e}_u|\!\| \leq Ch^k\,\|\boldsymbol{u}\|_{k+1,\Omega} + \|\!|\boldsymbol{\chi}_u|\!\|, \tag{4.10a}$$

$$\|e_\theta\|_h \leq Ch^k\,\|\theta\|_{k+1,\Omega} + \|\chi_\theta\|_h\,, \tag{4.10b}$$

$$\|e_p\|_{0,\Omega} \leq Ch^k\,\|p\|_{k,\Omega} + \|\chi_p\|_{0,\Omega}\,, \tag{4.10c}$$

To estimate the error it is therefore sufficient to estimate the terms $\|\!|\boldsymbol{\chi}_u|\!\|$, $\|\chi_\theta\|_h$ and $\|\chi_p\|_{0,\Omega}$.

We begin by proving the error in $(\boldsymbol{u},\theta)$. We note that the solution $(\boldsymbol{u},p,\theta)$ of the continuous problem (1.2) satisfies

$$\begin{aligned} A_h(\boldsymbol{u},\boldsymbol{v}_h)+O_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h)+B_h(\boldsymbol{v}_h,p) &= D_h(\theta,\boldsymbol{v}_h), \\ B_h(\boldsymbol{u},q_h) &= 0, \\ M_h(\theta,\psi_h)+C_h(\boldsymbol{u},\theta,\psi_h) &= F_h(\psi_h), \end{aligned}$$

for all $(\boldsymbol{v}_h,q_h,\psi_h)\in \boldsymbol{X}_h\times Q_h\times Z_h$. Subtracting the above equations from the discrete formulation (3.2), we obtain the error equations

$$A_h(\boldsymbol{e}_u,\boldsymbol{v}_h)+O_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h)-O_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)+B_h(\boldsymbol{v}_h,e_p)=D_h(e_\theta,\boldsymbol{v}_h), \tag{4.11a}$$

$$B_h(\boldsymbol{e}_u,q_h)=0, \tag{4.11b}$$

$$M_h(e_\theta,\psi_h)+C_h(\boldsymbol{u},\theta,\psi_h)-C_h(\boldsymbol{u}_h,\theta_h,\psi_h)=0, \tag{4.11c}$$

for all $(\boldsymbol{v}_h,q_h,\psi_h)\in \boldsymbol{X}_h\times Q_h\times Z_h$, which gives using (4.9) and (4.11a) with $\boldsymbol{v}_h=\boldsymbol{\chi}_u$

$$A_h(\boldsymbol{\chi}_u,\boldsymbol{\chi}_u)+O_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\chi}_u)-O_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{\chi}_u)+B_h(\boldsymbol{\chi}_u,e_p) \quad = \quad D_h(e_\theta,\boldsymbol{v}_h)-A_h(\boldsymbol{\eta}_u,\boldsymbol{\chi}_u).$$

Using again the decomposition (4.9), we have

$$\begin{aligned} O_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\chi}_u)-O_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{\chi}_u) &= O_h(\boldsymbol{e}_u,\boldsymbol{u},\boldsymbol{\chi}_u)+O_h(\boldsymbol{u}_h,\boldsymbol{e}_u,\boldsymbol{\chi}_u) \\ &= O_h(\boldsymbol{\eta}_u,\boldsymbol{u},\boldsymbol{\chi}_u)+O_h(\boldsymbol{\chi}_u,\boldsymbol{u},\boldsymbol{\chi}_u)+O_h(\boldsymbol{u}_h,\boldsymbol{\eta}_u,\boldsymbol{\chi}_u)+\underbrace{O_h(\boldsymbol{u}_h,\boldsymbol{\chi}_u,\boldsymbol{\chi}_u)}_{=0}, \end{aligned}$$

Oberve that from the definition of I_h in (4.1)-(4.2), we have $B_h(\boldsymbol{\chi}_u, \chi_p) = 0$ and then

$$B_h(\boldsymbol{\chi}_u, e_p) = B_h(\boldsymbol{\chi}_u, \eta_p) + \underbrace{B_h(\boldsymbol{\chi}_u, \chi_p)}_{=0}$$

Hence, we get

$$\begin{array}{rcl} A_h(\boldsymbol{\chi}_u, \boldsymbol{\chi}_u) & = & D_h(e_\theta, \boldsymbol{\chi}_u) - A_h(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) - B_h(\boldsymbol{\chi}_u, \eta_p) - O_h(\boldsymbol{\eta}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) \\ & - & O_h(\boldsymbol{\chi}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) - O_h(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_u). \end{array}$$

Using Lemmas 3.1, Lemma 3.3 and Lemma 3.4, we have

$$\begin{array}{rcl} \nu\alpha_{A_h} |||\boldsymbol{\chi}_u|||^2 & \leq & C_{D_h} \|\boldsymbol{g}\|_{0,\Omega} \|e_\theta\|_h \; |||\boldsymbol{\chi}_u||| + \nu C_{A_h} |||\boldsymbol{\eta}_u||| \; |||\boldsymbol{\chi}_u||| + C_{B_h}\|\eta_p\|_{0,\Omega}|||\boldsymbol{\chi}_u||| + C_{O_h}|||\boldsymbol{\eta}_u||| \; |||\boldsymbol{u}||| \; |||\boldsymbol{\chi}_u||| \\ & + & C_{O_h}|||\boldsymbol{u}||| \; |||\boldsymbol{\chi_u}|||^2 + C_{O_h}|||\boldsymbol{u}_h||| \; |||\boldsymbol{\eta}_u||| \, |||\boldsymbol{\chi}_u|||. \end{array}$$

Now, we recall that thanks to the *a priori* estimates (2.32) and (3.23), we have

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq K_1(\nu, \kappa, \boldsymbol{g}, f) \quad \text{and} \quad |||\boldsymbol{u}_h||| \leq C_1(\nu, \kappa, \boldsymbol{g}, f),$$

which implies

$$\begin{aligned} \Big(\nu\alpha_{A_h} - C_{O_h} K_1(\nu, \kappa, \boldsymbol{g}, f)\Big)|||\boldsymbol{\chi}_u||| \leq C_{D_h} \; \|\boldsymbol{g}\|_{0,\Omega} \|\chi_\theta\|_h + C_{D_h} \; \|\boldsymbol{g}\|_{0,\Omega} \|\eta_p\|_{0,\Omega} + \nu C_{A_h} |||\boldsymbol{\eta}_u||| \\ + C_{B_h}\|\eta_p\|_{0,\Omega} + C_{O_h}\big(K_1(\nu, \kappa, \boldsymbol{g}, f) + C_1(\nu, \kappa, \boldsymbol{g}, f)\big)|||\boldsymbol{\eta}_u|||. \end{aligned}$$

So, we have

$$\Big(\nu\alpha_{A_h} - C_{O_h} K_1(\nu, \kappa, \boldsymbol{g}, f)\Big)|||\boldsymbol{\chi}_u||| \leq C_{D_h} \; \|\boldsymbol{g}\|_{0,\Omega} \|\chi_\theta\|_h + C_* \tag{4.12}$$

with $C_* = C_{D_h} \|\boldsymbol{g}\|_{0,\Omega} \|\eta_p\|_{0,\Omega} + \nu C_{A_h} |||\boldsymbol{\eta}_u||| + C_{B_h}\|\eta_p\|_{0,\Omega} + C_{O_h}\big(K_1(\nu, \kappa, \boldsymbol{g}, f) + C_1(\nu, \kappa, \boldsymbol{g}, f)\big)|||\boldsymbol{\eta}_u|||$.
To estimate $\|\chi_\theta\|_h$, we use (4.11c) with $\psi_h = \chi_\theta$

$$M_h(e_\theta, \chi_\theta) + C_h(\boldsymbol{u}, \theta, \chi_\theta) - C_h(\boldsymbol{u}_h, \theta_h, \chi_\theta) \quad = \quad 0,$$

which gives using the error decompositions (4.9) and the positivity of the form C_h (see (3.15)):

$$M_h(\chi_\theta, \chi_\theta) + \underbrace{C_h(\boldsymbol{u}_h, \chi_\theta, \chi_\theta)}_{\geq 0} \quad = \quad -M_h(\eta_\theta, \chi_\theta) - C_h(\boldsymbol{\eta}_u, \theta, \chi_\theta) - C_h(\boldsymbol{\chi}_u, \theta, \chi_\theta) - C_h(\boldsymbol{u}_h, \eta_\theta, \chi_\theta).$$

Using again the *a priori* estimate for $\boldsymbol{u}_h$ in (3.23) and that for θ : (see (2.32)):

$$\|\theta\|_{\boldsymbol{H}^1(\Omega)} \leq K_2(\kappa, f),$$

together with the continuity of the forms M_h and C_h (see (3.9b) and (3.13c)), we obtain:

$$\begin{aligned} \kappa\, \alpha_{M_h} \|\chi_\theta\|_h &\leq C_{M_h} \|\eta_\theta\|_h + C_{C_h} |||\boldsymbol{\eta}_u||| \; \|\theta\|_h + C_{C_h} |||\boldsymbol{\chi}_u||| \; \|\theta\|_h + C_{C_h} |||\boldsymbol{u}_h||| \; \|\eta_\theta\|_h \\ &\leq C_{M_h} \|\eta_\theta\|_h + C_{C_h} K_2(\kappa, f)\big(|||\boldsymbol{\eta}_u||| + |||\boldsymbol{\chi}_u|||\big) + C_{C_h} C_1(\nu, \kappa, \boldsymbol{g}, f) \; \|\eta_\theta\|_h \\ &\leq C_{C_h} K_2(\kappa, f)|||\boldsymbol{\chi}_u||| + \widetilde{C}_*, \end{aligned} \tag{4.13}$$

with

$$\widetilde{C}_* = C_{M_h} \|\eta_\theta\|_h + C_{C_h} K_2(\kappa, f)|||\boldsymbol{\eta}_u||| + C_{C_h} C_1(\nu, \kappa, \boldsymbol{g}, f) \; \|\eta_\theta\|_h \,.$$

Combining the estimates (4.12) and (4.13), we obtain

$$\Big(\nu\alpha_A - C_{O_h} K_1(\nu, \kappa, \boldsymbol{g}, f) - \frac{C_{D_h} \|\boldsymbol{g}\|_{0,\Omega}}{\kappa\alpha_{M_h}}\Big)|||\boldsymbol{\chi}_u||| \quad \leq \quad \frac{C_{D_h} \|\boldsymbol{g}\|_{0,\Omega}}{\kappa\alpha_{M_h}}\widetilde{C}_* + C_*,$$

which gives by using assumption (4.6)

$$\frac{1}{2}\nu\alpha_A|||\boldsymbol{\chi}_u||| \quad \leq \quad \frac{C_{D_h}\,\|\boldsymbol{g}\|_{0,\Omega}}{\kappa\alpha_{M_h}}\widetilde{C}_* + C_*. \tag{4.14}$$

Replacing (4.14) in (4.13) gives

$$\kappa\,\alpha_{M_h}\,\|\chi_\theta\|_h \leq \frac{2C_{C_h}K_2(\kappa,f)}{\nu\alpha_A}\Big(\frac{C_{D_h}\,\|\boldsymbol{g}\|_{0,\Omega}}{\kappa\alpha_{M_h}}\widetilde{C}_* + C_*\Big) + \widetilde{C}_* \tag{4.15}$$

The energy-norm error estimate for the velocity $\boldsymbol{u}_h$ and the temperature θ_h in (4.7) now follows by the bounds in (4.14),(4.15), (4.10a) and (4.10b).

To prove the error estimate on the pressure in (4.8), we use the inf-sup condition (3.8), the continuity property of B_h (3.9c) and the fact that $\chi_p = \eta_p - e_p$ to write

$$\begin{aligned}\beta\,\|\chi_p\|_{0,\Omega} \quad &\leq \quad \sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\frac{B_h(\boldsymbol{v}_h,\chi_p)}{|||\boldsymbol{v}_h|||} \\ &\leq \quad C_{B_h}\,\|\eta_p\|_{0,\Omega} + \sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\frac{B_h(\boldsymbol{v}_h,-e_p)}{|||\boldsymbol{v}_h|||}\end{aligned} \tag{4.16}$$

To bound the term $B_h(\boldsymbol{v}_h, -e_p)$, we first rewrite equation (4.11a), adding and subtracting suitable terms to obtain:

$$\begin{aligned}B_h(\boldsymbol{v}_h,-e_p) \quad &= \quad A_h(\boldsymbol{e}_u,\boldsymbol{v}_h) + O_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h) - O_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) - D_h(e_\theta,\boldsymbol{v}_h) \\ &= \quad A_h(\boldsymbol{e}_u,\boldsymbol{v}_h) + O_h(\boldsymbol{e}_u,\boldsymbol{u},\boldsymbol{v}_h) + O_h(\boldsymbol{u}_h,\boldsymbol{e}_u,\boldsymbol{v}_h) - D_h(e_\theta,\boldsymbol{v}_h)\end{aligned}$$

From the continuity properties of A_h, O_h and D_h (see (3.9a), (3.13b) and (3.13a)), we have

$$|B_h(\boldsymbol{v}_h,-e_p)| \quad \leq \quad \Big(C_{A_h} + C_{O_h}\,\|\boldsymbol{u}\|_{1,\Omega} + C_{O_h}|||\boldsymbol{u}_h|||\Big)|||\boldsymbol{e}_u|||\;|||\boldsymbol{v}_h||| + C_{D_h}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|e_\theta\|_h\,|||\boldsymbol{v}_h|||$$

By replacing this last bound into (4.16) and using the *a priori* estimate for $\boldsymbol{u}$ and $\boldsymbol{u}_h$, we get

$$\beta\,\|\chi_p\|_{0,\Omega} \leq\; C_{B_h}\,\|\eta_p\|_{0,\Omega} + \Big(C_{A_h} + C_{O_h}K_1(\nu,\kappa,\boldsymbol{g},f) + C_{O_h}C_1(\nu,\kappa,\boldsymbol{g},f)\Big)|||\boldsymbol{e}_u||| + C_{D_h}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|e_\theta\|_h$$

Using estimates (4.14) and (4.15) together with (4.10) implies the desired estimate (4.8) for the error on the pressure.

□

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