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Instationary Stokes problem with pressure boundary condition in L^p-spaces

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Abstract. In this paper, we prove the analyticity of the semi-group generated by the Stokes operator with a boundary condition involving the pressure. This allows us to obtain weak and strong solutions for the time-dependent Stokes problem with the corresponding boundary condition.

1. Introduction

We consider in a bounded cylindrical domain $\Omega \times (0, T)$ the instationary Stokes problem

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \times (0, T), \\ \qquad\qquad\qquad\qquad\quad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Problem (1.1) has often been studied with Dirichlet boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on Γ. However, in some real-life situations it is natural to prescribe the value of the pressure at least on some part of the boundary, and this can arise in case of pipelines, blood vessels and different hydraulic systems involving pumps. The well-posedness of the incompressible Stokes and Navier–Stokes equations with appropriate non-standard boundary conditions is a challenging problem, from both the theoretical and numerical point of view. It should be mentioned that there is no physical justification for prescribing only a pressure boundary condition on the boundary and it must be completed by adding some boundary condition involving the velocity. For example, we can consider the tangential part of the velocity on the boundary at the same time

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, \quad \pi = 0 \quad \text{on} \quad \Gamma \times (0, T). \tag{1.2}$$

The Stokes problem (1.1) with the boundary condition (1.2) has been studied in several papers. Conca et al. studied in [11,12] the stationary Stokes and Navier–Stokes systems, with boundary conditions involving the pressure in a bounded three-dimensional domain and got existence and uniqueness results. Precisely, the boundary

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conditions are assumed to be of three different types: the velocity is given in a portion of the boundary; the pressure and the tangential component of the velocity are given on a second part of the boundary; the normal component of the velocity and the tangential component of the vorticity are given on the remaining part of the boundary. Gresho [15] gave an overview of the state of the art in the nineties concerning the Stokes and Navier–Stokes problem subject to several boundary conditions for the velocity, pressure and vorticity. He also discussed time-dependent flows and gave methods for starting these flows in the context of numerical simulations. Gresho and Sanin [16] studied the standard pressure Poisson equation (PPE) equipped with a Neumann pressure boundary condition derived from the momentum equation in the normal direction. They formulated a hypothesis that the Navier–Stokes problem and the (PPE) would be equivalent. Pressure boundary conditions were also discussed in [18], where the authors Heywood et al. reviewed the mathematical formulations for unbounded domains for a class of problems that involve the prescription of pressure drops or net flux conditions. Emphasis was placed on the advantage of pressure boundary conditions for modeling and numerical simulations. Köhne [19] proved the maximal L^p regularity of the solutions to the Stokes and Navier–Stokes equations endowed with energy preserving boundary conditions. The prove is done using some results on the L^p theory for elliptic and parabolic problems. In [21], Lukaszewicz considered the initial Navier–Stokes problem in a class of space–time domains where a dynamic pressure is given on a part of the boundary and proved the existence of weak solutions. An incompressible Newtonian fluid in a finite number of pipes with impermeable walls is considered in [22]. Marušić [23] studied the non-stationary Navier–Stokes problem in a bounded domain of $\mathbb{R}^2$ where the boundary condition (1.2) is considered in a part of the boundary and proved the local existence and uniqueness of weak solutions. More recently, Bothe et al. [10] studied the incompressible Stokes and Navier–Stokes problems with a large class of non-standard boundary conditions; some of them were the boundary condition (1.2) and inhomogeneous pressure boundary conditions. They introduced first the concept of energy preserving boundary conditions. They established then an L^p-theory for the Stokes system under each of these boundary conditions and applied their results to the Navier–Stokes problem. Their results are not limited to the three space dimensions.

In this paper, we consider the linearized time-dependent Stokes problem (1.1) with the boundary condition (1.2). Our study of this problem is based on the semi-group theory. For this reason, we consider the complex resolvent of the Stokes operator with this boundary condition

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \pi = 0 \quad \text{on } \Gamma. \end{cases} \tag{1.3}$$

Due to boundary conditions (1.2), the pressure equation is decoupled from the solution of the velocity components. Indeed taking the divergence of the first equation in (1.1), the pressure is a solution of a Dirichlet problem. Assuming that $\operatorname{div} \boldsymbol{f} = 0$, π is equal to zero and we are reduced to study the following heat problem:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \times (0, T), \\ \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma \times (0, T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \end{cases} \tag{1.4}$$

The organization of the paper is as follows. In Sect. 2, we introduce most of the notations used in this work and collect some preliminary results. Section 3 deals with the analyticity of the semi-group generated by the Stokes operator with the pressure boundary condition (1.2). We study in Sect. 4 the existence of weak and strong solutions $(\boldsymbol{u}, \pi)$ of the time-dependent homogeneous Stokes problem (1.1)–(1.2). Finally in "Appendix," we prove the existence of a trace operator in a subspace of $W^{-1,p}(\Omega)$ to be determined.

2. Notations and preliminary results

In this section, we give some notations and basic definitions and collect several known results which are going to be useful for us in the sequel.

In what follows, if we do not state otherwise, Ω will be considered as an open bounded domain of $\mathbb{R}^3$ of class $C^{2,1}$. In some situation, we suppose that Ω is of class $C^{1,1}$ in the case where the regularity $C^{1,1}$ is sufficient for the proof. A unit normal vector to the boundary can be defined almost everywhere, and it will be denoted by $\boldsymbol{n}$. The generic point in Ω is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$.

We do not assume that the boundary Γ is connected and we denote by Γ_i, $0 \leq i \leq I$, the connected component of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Omega}$. We also fix a smooth open set ϑ with a connected boundary (a ball, for instance), such that $\overline{\Omega}$ is contained in ϑ, and we denote by Ω_i, $0 \leq i \leq I$, the connected component of $\vartheta \backslash \overline{\Omega}$ with boundary Γ_i ($\Gamma_0 \cup \partial\vartheta$ for $i = 0$).

Finally, vector fields, matrix fields and their corresponding spaces defined on Ω will be denoted by bold character. The functions treated here are complex-valued functions. We will use also the symbol σ to represent a set of divergence free functions.

Now, we introduce some function spaces. First, we denote by $\boldsymbol{\mathcal{D}(\Omega)}$ the set of infinitely differentiable functions with compact support in Ω and by $\boldsymbol{\mathcal{D}(\overline{\Omega})}$ the restriction to Ω of infinitely differentiable functions with compact support in $\mathbb{R}^3$. We denote also by $\boldsymbol{L}^p(\Omega)$ the usual vector-valued $\boldsymbol{L}^p$-space over Ω. For any $1 < p < \infty$, we define the space:

$$\boldsymbol{L}^p_\sigma(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega \right\},$$

which is a closed subspace of $\boldsymbol{L}^p(\Omega)$ equipped with the norm of $\boldsymbol{L}^p(\Omega)$. We also consider the space

$$\boldsymbol{H}^p(\mathbf{curl}, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \right\},$$

equipped with the graph norm. We know that (cf. [8, Section 2] and [5, Proposition 2.3]), for every $1 < p < \infty$, $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ and for any function $\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega)$ the tangential boundary value $\boldsymbol{v} \times \boldsymbol{n}_{|\Gamma}$ of $\boldsymbol{v}$ exists and belongs to $\boldsymbol{W}^{-1/p,\,p}(\Gamma)$. More precisely, for all function $\boldsymbol{v}$ in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ has a tangential trace $\boldsymbol{v} \times \boldsymbol{n}$ in $\boldsymbol{W}^{-1/p,\,p}(\Gamma)$ defined by:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}^{1,\,p'}(\Omega), \quad \langle \boldsymbol{v} \times \boldsymbol{n},\, \boldsymbol{\varphi}\rangle_\Gamma \;=\; \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \overline{\boldsymbol{\varphi}}\, \mathrm{d}\, x \;-\; \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \overline{\boldsymbol{\varphi}}\, \mathrm{d}\, x$$

with $\langle .,\ .\rangle_\Gamma$ the anti-duality between $\boldsymbol{W}^{-1/p,\,p}(\Gamma)$ and $\boldsymbol{W}^{1/p,\,p'}(\Gamma)$. We consider similarly the space

$$\boldsymbol{H}^p(\mathrm{div}, \Omega) \;=\; \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\;\; \mathrm{div}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\right\},$$

equipped with the graph norm. For every $1 < p < \infty$, $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ (cf. [8, Section 2] and [5, Proposition 2.3]). Furthermore, for any function $\boldsymbol{v}$ in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ the normal trace $\boldsymbol{v} \cdot \boldsymbol{n}_{|\Gamma}$ exists and belongs to $W^{-1/p,\,p}(\Gamma)$. Moreover, we have:

$$\forall \varphi \in W^{1,\,p'}(\Omega), \quad \langle \boldsymbol{v} \cdot \boldsymbol{n},\, \varphi\rangle_\Gamma \;=\; \int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\, \overline{\varphi}\, \mathrm{d}\, x \;+\; \int_\Omega (\mathrm{div}\, \boldsymbol{v})\, \overline{\varphi}\, \mathrm{d}\, x,$$

where $\langle .,\ .\rangle_\Gamma$ is the anti-duality between between $W^{-1/p,\,p}(\Gamma)$ and $W^{1/p,\,p'}(\Gamma)$.

As described in [8, Section 2], the adherence of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ is, respectively, equal to

$$\begin{aligned} \boldsymbol{H}^p_0(\mathbf{curl}, \Omega) \;&=\; \left\{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega);\;\; \boldsymbol{v} \times \boldsymbol{n} \;=\; \boldsymbol{0} \text{ on } \Gamma\right\},\\ \boldsymbol{H}^p_0(\mathrm{div}, \Omega) \;&=\; \left\{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div}, \Omega);\;\; \boldsymbol{v} \cdot \boldsymbol{n} \;=\; 0 \text{ on } \Gamma\right\}. \end{aligned}$$

Next we consider the subspaces

$$\boldsymbol{X}^p_N(\Omega) \;=\; \left\{\boldsymbol{v} \in \boldsymbol{H}^p_0(\mathbf{curl},\ \Omega);\;\mathrm{div}\, \boldsymbol{v} \in L^p(\Omega)\right\}$$

and

$$\boldsymbol{X}^p_{N,\sigma}(\Omega) \;=\; \left\{\boldsymbol{v} \in \boldsymbol{X}^p_N(\Omega);\;\mathrm{div}\, \boldsymbol{v} = 0 \;\text{ in } \Omega\right\}, \tag{2.1}$$

which are Banach spaces for the graph norm of $\boldsymbol{X}^p_N(\Omega)$. Thanks to [8, Theorem 3.2], since Ω is in particular of class $C^{1,1}$, the space $\boldsymbol{X}^p_N(\Omega)$ is continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$ and

$$\forall\, \boldsymbol{v} \in \boldsymbol{X}^p_N(\Omega), \quad \|\boldsymbol{v}\|_{\boldsymbol{X}^p_N(\Omega)} \simeq \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

The dual spaces $[\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)]'$ and $[\boldsymbol{H}^p_0(\mathrm{div}, \Omega)]'$ of $\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}^p_0(\mathrm{div}, \Omega)$, respectively, can be characterized as follows (cf. [25, Proposition 1.0.5, Proposition 1.0.6]): A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)]'$ if and only if there exist functions $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{\xi} \in \boldsymbol{L}^{p'}(\Omega)$, such that $\boldsymbol{f} \;=\; \boldsymbol{\psi} \;+\; \mathbf{curl}\, \boldsymbol{\xi}$. Moreover, one has

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'} \;=\; \inf_{\boldsymbol{f} \,=\, \boldsymbol{\psi} \,+\, \mathbf{curl}\, \boldsymbol{\xi}} \max\,(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)},\; \|\boldsymbol{\xi}\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

Similarly, a distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^p(\mathrm{div},\Omega)]'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\chi \in L^{p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\,\chi$ and

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^p(\mathrm{div},\Omega)]'} = \inf_{\boldsymbol{f}=\boldsymbol{\psi}+\mathbf{grad}\,\chi} \max\left(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)},\ \|\chi\|_{\boldsymbol{L}^{p'}(\Omega)}\right).$$

Finally, we define also the space

$$\boldsymbol{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}_\sigma^p(\Omega);\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}. \tag{2.2}$$

Thanks to [5, Proposition 3.18] and to [8, Corollary 4.2] we know that the space $\boldsymbol{K}_N^p(\Omega)$ is independent of p, is of finite dimension and is spanned by the functions ∇q_i^N, $i = 1, \ldots, I$, where q_i^N is the unique solution in $W^{2,p}(\Omega)$ of the problem

$$\begin{cases} -\Delta q_i^N = 0 \quad \text{ in } \Omega, \\ q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^N|_{\Gamma_k} = \text{constant}, \quad 1 \le k \le I, \\ \left\langle \partial_n q_i^N,\ 1\right\rangle_{\Gamma_k} = \delta_{i\,k},\ \ 1 \le k \le I, \quad \text{and} \quad \left\langle \partial_n q_i^N,\ 1\right\rangle_{\Gamma_0} = -1 \end{cases}$$

In the rest of this section, we will give some definitions and preliminary results which are essential in our work. First, we recall that (see, e.g., [8, Lemma 5.2] and [2, Corollary 2.2.13]) for any vector $\boldsymbol{v}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that div $\boldsymbol{v}$ in $W^{1,p}(\Omega)$ one has:

$$\mathrm{div}\,\boldsymbol{v} = \mathrm{div}_\Gamma\,\boldsymbol{v}_{\boldsymbol{\tau}} + 2\,K\,\boldsymbol{v}\cdot\boldsymbol{n} + \frac{\partial\,\boldsymbol{v}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} \quad \text{in } W^{-\frac{1}{p},p}(\Gamma), \tag{2.3}$$

where $K = \frac{1}{2}\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{n}}{\partial\,s_j}\cdot\boldsymbol{\tau_j}$ denotes the mean curvature and div_Γ is the surface divergence.

We recall also the following lemma that plays an important role in the proof of the resolvent estimates in L^p theory (see [3, Lemma 2.4] for the proof):

LEMMA 2.1. *Suppose that Ω is of class $C^{1,1}$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then, for all $p \geq 2$ one has:*

$$\begin{aligned} -\int_\Omega |\boldsymbol{u}|^{p-2}\Delta\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x &= \int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega \left|\nabla|\boldsymbol{u}|^{p/2}\right|^2\mathrm{d}\,x \\ &+ (p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,Re\left(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\right)\mathrm{Im}\left(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\right)\mathrm{d}\,x - \left\langle\frac{\partial\,\boldsymbol{u}}{\partial\boldsymbol{n}},\ |\boldsymbol{u}|^{p-2}\boldsymbol{u}\right\rangle_\Gamma, \end{aligned}$$

where $\langle .\,,\ .\rangle_\Gamma$ is the anti-duality between $\mathbf{W}^{-1/p,p}(\Gamma)$ and $\mathbf{W}^{1/p,p'}(\Gamma)$.

Now we want to define the Stokes operator with the boundary conditions described above in the context of L^p-spaces, $1 < p < \infty$. Let $\boldsymbol{u} \in \boldsymbol{L}_\sigma^p(\Omega)$ be fixed and consider the mapping

$$\begin{aligned} A_p\boldsymbol{u}:\quad &\boldsymbol{W} \longrightarrow \mathbb{C} \\ &\boldsymbol{v} \longrightarrow -\int_\Omega \boldsymbol{u}\cdot\Delta\overline{\boldsymbol{v}}\,\mathrm{d}\,x, \end{aligned}$$

where

$$\boldsymbol{W} = \boldsymbol{X}^{p'}_{N,\sigma}(\Omega) \cap \boldsymbol{W}^{2,p'}(\Omega).$$

(We recall that $\boldsymbol{X}^{p}_{N,\sigma}(\Omega)$ is given by (2.1).) It is clear that $A_p \in \mathcal{L}(\boldsymbol{L}^p_\sigma(\Omega),\ \boldsymbol{W}')$. Let $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ and suppose that $A_p\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$, using de Rham's Lemma there exists $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$ such that

$$A_p\boldsymbol{u} + \Delta\boldsymbol{u} = \nabla\pi \quad \text{in} \quad \Omega.$$

Notice that the pressure π is in $W^{-1,p}(\Omega)$ (the dual of $W^{1,p'}_0(\Omega)$) and satisfies $\Delta\pi = 0$ in Ω. We know that (see [20, Theorem 6.5] for $p = 2$ and "Appendix" below for $p \neq 2$) the trace value $\pi_{|\Gamma}$ is well defined and belongs to $W^{-1-1/p,p}(\Gamma)$. This means that if $\pi = 0$ on Γ, $\pi = 0$ in Ω and $A_p\boldsymbol{u} = -\Delta\boldsymbol{u}$ in Ω. Next, observe that since $\Delta\boldsymbol{u} = -A_p\boldsymbol{u}$, then thanks to [8, Theorem 5.4], we know that $\boldsymbol{u}\times\boldsymbol{n}_{|\Gamma} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$. Moreover, if we suppose that $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ, then $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ is a solution of the problem

$$\begin{cases} -\Delta\boldsymbol{u} = A_p\boldsymbol{u}, & \operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \ \Omega, \\ & \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on} \ \Gamma. \end{cases}$$

Thus, using the regularity result in [8, Corollary 5.4] we deduce that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ since Ω is of class $C^{2,1}$.

Then, the operator $A_p : \mathbf{D}(A_p) \subset \boldsymbol{L}^p_\sigma(\Omega) \longmapsto \boldsymbol{L}^p_\sigma(\Omega)$ is a linear operator that can be characterized as follows

$$\mathbf{D}(A_p) = \Big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}, \ \text{ on } \Gamma\Big\}. \tag{2.4}$$

Moreover,

$$\forall\ \boldsymbol{u} \in \mathbf{D}(A_p), \quad A_p\boldsymbol{u} = -\Delta\boldsymbol{u} \quad \text{in} \quad \Omega. \tag{2.5}$$

In the rest of this paper, we consider the Stokes operator A_p defined in $\boldsymbol{L}^p_\sigma(\Omega)$. The following proposition shows the density of the domain of the Stokes operator.

PROPOSITION 2.2. *The domain* $\mathbf{D}(A_p)$ *defined in* (2.4) *is dense in* $\boldsymbol{L}^p_\sigma(\Omega)$.

Proof. First observe that since $\boldsymbol{\mathcal{D}}(\boldsymbol{\Omega})$ is dense in $\boldsymbol{L}^p(\Omega)$, then the space

$$\big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\big\}$$

is dense in $\boldsymbol{L}^p(\Omega)$. Let $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ and let $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ be a sequence in $\boldsymbol{W}^{2,p}(\Omega)$ such that for all k in $\mathbb{N}$, we have $\boldsymbol{u}_k\times\boldsymbol{n} = \boldsymbol{0}$ on Γ and the sequence $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$. Now, for all k in $\mathbb{N}$, we consider the unique solution $\chi_k \in W^{3,p}(\Omega)\cap W^{1,p}_0(\Omega)$ of the Problem

$$\Delta\chi_k = \operatorname{div}\boldsymbol{u}_k \ \text{ in } \ \Omega \quad \text{and} \quad \chi_k = 0 \quad \text{on} \ \ \Gamma.$$

(see [14, Theorem 9.15 page 241, Theorem 9.19 page 243] and [17, Theorem 2.4.2.7 page 126, Theorem 2.5.1.1 page 128] for the regularity of solutions to elliptic problems).

The unique solution χ_k satisfies the estimate

$$\forall\, k \in \mathbb{N}, \quad \|\chi_k\|_{W^{1,p}(\Omega)} \;\leq\; C\, \|\mathrm{div}\, \boldsymbol{u}_k\|_{W^{-1,p}(\Omega)},$$

for some constant $C > 0$. Observe that since $(\mathrm{div}\, \boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to zero in $W^{-1,p}(\Omega)$, then $(\chi_k)_{k\in\mathbb{N}}$ converges to zero in $W^{1,p}(\Omega)$. Finally, for all $k \in \mathbb{N}$, we set

$$\boldsymbol{\varphi}_k \;=\; \boldsymbol{u}_k - \mathbf{grad}\, \chi_k.$$

We can easily verify that for every $k \in \mathbb{N}$, $\boldsymbol{\varphi}_k \in \boldsymbol{W}^{2,p}(\Omega)$, $\mathrm{div}\, \boldsymbol{\varphi}_k = 0$ in Ω, $\boldsymbol{\varphi}_k \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and that the sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$. This ends the proof. $\square$

The following proposition gives the closedness of the Stokes operator.

PROPOSITION 2.3. *The operator A_p is a closed operator.*

Proof. Let $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbf{D}(A_p)$ that converges to a function $\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$ and $(\Delta \boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to a function $\boldsymbol{y}$ in $\boldsymbol{L}^p(\Omega)$. Then, $(\mathrm{div}\, \boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to $\mathrm{div}\, \boldsymbol{u}$ in $W^{-1,p}(\Omega)$ and since for all $k \in \mathbb{N}$, $\mathrm{div}\, \boldsymbol{u}_k = 0$ in Ω, then $\mathrm{div}\, \boldsymbol{u} = 0$ in Ω. Moreover, $(\Delta \boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to $\Delta \boldsymbol{u}$ in $\boldsymbol{W}^{-2,p}(\Omega)$ and $(\Delta \boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{y}$ in $\boldsymbol{L}^p(\Omega)$ (in particular in $\boldsymbol{W}^{-2,p}(\Omega)$). Thus, $\boldsymbol{y} = \Delta \boldsymbol{u}$ and $\Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$.

On the other hand, since $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, then using [7, Lemma 4] we obtain that $\boldsymbol{u}_{|\Gamma}$ is well defined in $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and $(\boldsymbol{u}_{k|\Gamma})_{k\in\mathbb{N}}$ converges to $\boldsymbol{u}_{|\Gamma}$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$. In addition, $(\boldsymbol{u}_k \cdot \boldsymbol{n})_k$ converges to $\boldsymbol{u} \cdot \boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$. As a result, $(\boldsymbol{u}_k \times \boldsymbol{n})_{k\in\mathbb{N}}$ converges to $\boldsymbol{u} \times \boldsymbol{n}$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$. Since for all $k \in \mathbb{N}$, $\boldsymbol{u}_k \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, then $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ.

It remains to prove that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. To this end, we show that $\boldsymbol{y} = \Delta \boldsymbol{u}$ satisfy the compatibility condition

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \quad \langle \boldsymbol{y}\,,\, \boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times \boldsymbol{L}^{p'}(\Omega)} = 0. \tag{2.6}$$

For all k in $\mathbb{N}$ and for all $\boldsymbol{v}$ in $\boldsymbol{K}_N^{p'}(\Omega)$, we have

$$\begin{aligned} \langle \Delta \boldsymbol{u}_k\,,\, \boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times \boldsymbol{L}^{p'}(\Omega)} &= -\langle \mathbf{curl}\,\mathbf{curl}\; \boldsymbol{u}_k\,,\, \boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times \boldsymbol{L}^{p'}(\Omega)} \\ &= \int_\Omega \mathbf{curl}\, \boldsymbol{u}_k \cdot \mathbf{curl}\, \bar{\boldsymbol{v}}\, \mathrm{d}x \;+\; \langle \boldsymbol{v} \times \boldsymbol{n}\,,\, \mathbf{curl}\, \boldsymbol{u}_k\rangle_\Gamma \\ &= 0. \end{aligned} \tag{2.7}$$

Finally since $(\Delta \boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{y} = \Delta \boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$, one has for every $\boldsymbol{v}$ in $\boldsymbol{K}_N^{p'}(\Omega)$

$$\langle \boldsymbol{y}\,,\, \boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times \boldsymbol{L}^{p'}(\Omega)} = \lim_{k\to+\infty} \langle \Delta \boldsymbol{u}_k\,,\, \boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times \boldsymbol{L}^{p'}(\Omega)} = 0.$$

In other words, $\boldsymbol{y}$ belongs to $\boldsymbol{L}^p(\Omega)$ and satisfy the compatibility condition (2.6); then, using [8, Corollary 5.4] we deduce that the function $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ since Ω is of class to $C^{2,1}$. All in all $\boldsymbol{u}$ belongs to $\mathbf{D}(A_p)$ and $\boldsymbol{y} = \Delta \boldsymbol{u}$, which ends the proof. $\square$

We end this section with some relevant properties of sectorial operators very useful in our work. Let X be a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \mapsto X$ a closed linear densely defined operator, where $D(\mathcal{A})$ equipped with the graph norm forms a Banach space. We consider also the sector

$$\Sigma_\theta = \Big\{\lambda \in \mathbb{C}^*; \quad |\arg \lambda| < \pi - \theta\Big\},$$

where $0 \leq \theta < \pi/2$ and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with $\mathbb{C}$ denotes the set of all complex numbers. Thanks to the work of Engel and Nagel [13, Chapter 2, page 96, Definition 4.1], we know that a linear densely defined operator $\mathcal{A}$ is sectorial if there exists a constant $M > 0$ and an angle $0 \leq \theta < \pi/2$ such that

$$\forall\, \lambda \in \Sigma_\theta, \quad \|R(\lambda,\, \mathcal{A})\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}, \tag{2.8}$$

where $R(\lambda,\, \mathcal{A}) = (\lambda\, I - \mathcal{A})^{-1}$. In other words, the resolvent of a sectorial operator contains a sector Σ_θ for some $0 \leq \theta < \pi/2$ and for every $\lambda \in \Sigma_\theta$ one has estimate (2.8). Moreover, we know that (see [13, Chapter 2, Theorem 4.6, page 101]) an operator $\mathcal{A}$ generates a bounded analytic semi-group if and only if $\mathcal{A}$ is sectorial in the previous sense. Using the work of Yosida [26], one can verify that the previous definition of a sectorial operator is equivalent to the one stated in the book of Barbu [9, Chapter 1, Theorem 3.2, page 30]. According to Barbu, a closed linear densely defined operator $\mathcal{A}$ is sectorial if its resolvent set contains the half plane $\{\lambda \in \mathbb{C}^*;\ Re\,\lambda \geq 0\}$, such that

$$\forall\, \lambda \in \mathbb{C}^*, \quad Re\,\lambda \geq 0, \quad \|R(\lambda,\, \mathcal{A})\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}.$$

We note that Engel and Nagel pointed out in [13, Comment 4.11, page 107] that the density of the domain of $\mathcal{A}$ is not needed. They also mentioned that a non-densely defined sectorial operator generates an analytic semi-group without the strong continuity property.

It is also important to know that the closedness property is deduced directly from the sectorial property of the operator. This is due to the fact that an operator with non-empty resolvent set $\rho(\mathcal{A})$ is closed.

3. Analyticity results

In this section, we prove the analyticity of the Stokes semi-group with normal and pressure boundary conditions on the spaces $\boldsymbol{L}^p_\sigma(\Omega)$ and $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'_\sigma$ for all $1 < p < \infty$. For this reason, we will study the following complex resolvent problem: Find $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ solution to

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ & \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \end{cases} \tag{3.1}$$

if $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'_\sigma$ (respectively $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ if $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$).

REMARK 3.1. (i) At first glance, one may think that Problem (3.1) is underdetermined since we have a boundary condition involving only the tangential component of $\boldsymbol{u}$ on Γ and we don't impose any further condition on the normal component. In reality, the condition $\operatorname{div} \boldsymbol{u} = 0$ in Ω gives us an additional information on some components of $\boldsymbol{u}$ and $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}$. In fact, it can be seen formally that using (2.3) one has the additional boundary condition

$$2\,K\,\boldsymbol{u}\cdot\boldsymbol{n} + \frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma,$$

in the sense of $W^{-\frac{1}{p},p}(\Gamma)$ (or in $W^{1-\frac{1}{p},p}(\Gamma)$). So that under some regularity assumptions on $\boldsymbol{u}$ we can prove that Problem (3.1) is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ solution to

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma \\ K\,\boldsymbol{u}\cdot\boldsymbol{n} + \frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases}$$

(ii) Notice that Problem (3.1) is also equivalent to the following problem: Find $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ solution to

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f} & \text{in} \quad \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}, \quad \operatorname{div}\boldsymbol{u} = 0 & \text{on} \quad \Gamma. \end{cases} \tag{3.2}$$

Indeed, let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem (3.2) and set $\chi = \operatorname{div}\boldsymbol{u}$, the condition $\operatorname{div}\boldsymbol{f} = 0$ in Ω implies that

$$\begin{cases} \lambda\chi - \Delta\chi = 0 & \text{in} \quad \Omega, \\ \chi = 0 & \text{on} \quad \Gamma. \end{cases}$$

Thus, $\operatorname{div}\boldsymbol{u} = \chi = 0$ in Ω. Note that the method used in [10, Section 3.2] and [19, Section 4.5] leads also to the perception that the divergence constraint should be prescribed in the boundary of the domain and not in the interior.
(iii) Usually when we impose, the constraint $\operatorname{div}\boldsymbol{u} = 0$ in Ω appears a gradient of pressure in Problem (3.1). However as described in the previous section, due to the boundary conditions (1.2) the pressure can be decoupled from the system (1.3) by solving the following Dirichlet problem

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in}\,\Omega \quad \text{and} \quad \pi = 0 \text{ on } \Gamma. \tag{3.3}$$

Observe that if $\operatorname{div}\boldsymbol{f} = 0$ in Ω, the pressure π is zero. For this reason, the work is reduced to study the Problem (3.1) which is a resolvent problem for the Laplacian with $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$. Then, for a function $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ we recover the pressure by solving the Dirichlet Problem (3.3).

In what follows, we will use the following formula of the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$:

$$\Delta\,\boldsymbol{v} = \mathbf{grad}\,(\operatorname{div}\boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}.$$

The following theorem gives the solution of the resolvent Problem (3.1) in an L^2 context as well as a resolvent estimate. Before we state our theorem, we recall the following lemma.

LEMMA 3.2. *Let* $\varepsilon \in\,]0, \pi[$ *be fixed and let* Σ_ε *be the sector*

$$\Sigma_\varepsilon = \big\{\lambda \in \mathbb{C}^*;\ |\arg \lambda| \leq \pi - \varepsilon\big\}.$$

There exists a constant $C_\varepsilon > 0$ *such that for every positive real numbers a and b one has:*

$$\forall\, \lambda \in \Sigma_\varepsilon, \quad |\lambda\, a + b| \geq C_\varepsilon(|\lambda| a + b).$$

Now we can state and prove our theorem.

THEOREM 3.3. *Let* $\varepsilon \in\,]0, \pi[$ *be fixed and* $\lambda \in \Sigma_\varepsilon$ *and let* $\boldsymbol{f} \in [\boldsymbol{H}_0^2(\mathbf{curl}, \Omega)]'_\sigma$. *The Problem (3.1) has a unique solution* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. *Moreover, if* $\boldsymbol{f} \in \boldsymbol{L}^2_\sigma(\Omega)$, *then* $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, *and it satisfies the estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_\varepsilon}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{3.4}$$

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_\varepsilon}{\sqrt{|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.5}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq \frac{C(\Omega, \lambda, \varepsilon)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{3.6}$$

for some constant $C_\varepsilon > 0$ *and* $C(\Omega, \lambda, \varepsilon) = C(\Omega)(C_\epsilon + 1)(1 + |\lambda|)$.

Proof. The proof is done in three steps:
(i) Variational formulation: Consider the variational problem: Find $\boldsymbol{u} \in \boldsymbol{X}^2_{N,\sigma}(\Omega)$

$$\forall\, \boldsymbol{v} \in \boldsymbol{X}^2_{N,\sigma}(\Omega), \quad a(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}\,,\ \boldsymbol{v}\rangle_\Omega, \tag{3.7}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \lambda \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}}\,\mathrm{d}x$$

and

$$\langle .\,,\ .\rangle_\Omega = \langle .\,,\ .\rangle_{[\boldsymbol{H}_0^2(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^2(\mathbf{curl},\Omega)}.$$

Using Lemma 3.2, we can easily verify that a is a continuous coercive sesqui-linear form on $\boldsymbol{X}^2_{N,\sigma}(\Omega)$. Thus, due to Lax–Milgram Lemma Problem (3.7) has a unique solution $\boldsymbol{u} \in \boldsymbol{X}^2_{N,\sigma}(\Omega)$ since the right-hand side belongs to the anti-dual $(\boldsymbol{X}^2_{N,\sigma}(\Omega))'$.
(ii) Equivalent problem: Problem (3.7) can be extended to any test function $\boldsymbol{v} \in \boldsymbol{X}^2_N(\Omega)$. Indeed, let $\boldsymbol{v} \in \boldsymbol{X}^2_N(\Omega)$ and consider the unique solution $\chi \in H^2(\Omega) \cap H^1_0(\Omega)$ of the problem

$$\Delta\chi = \operatorname{div}\boldsymbol{v} \ \text{ in } \ \Omega, \quad \chi = 0 \ \text{ on } \ \Gamma.$$

Setting $\boldsymbol{\varphi} = \boldsymbol{v} - \mathbf{grad}\,\chi$, we can easily verify that $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$, $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω, $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{\varphi} \times \boldsymbol{n} = \mathbf{0}$ on Γ (i.e., $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{N,\sigma}(\Omega)$). Observe that $\mathbf{grad}\,\chi$ belongs to $\boldsymbol{H}^2_0(\mathbf{curl},\Omega)$, and using the density of $\mathcal{D}(\Omega)$ in $H^1_0(\Omega)$ we can easily verify that $\langle \boldsymbol{f}\,,\,\mathbf{grad}\,\chi\rangle_\Omega = 0$. As a result

$$\langle \boldsymbol{f}\,,\,\boldsymbol{v}\rangle_\Omega = \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}\rangle_\Omega.$$

We recall that $\langle .\,,\,.\rangle_\Omega = \langle .\,,\,.\rangle_{[\boldsymbol{H}^2_0(\mathbf{curl},\Omega)]'\times\boldsymbol{H}^2_0(\mathbf{curl},\Omega)}$.
On the other hand since $\operatorname{div}\boldsymbol{u} = 0$ in Ω, we can easily verify that

$$\int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}x = \int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}x.$$

As a consequence, we obtain $a(\boldsymbol{u},\boldsymbol{v}) = a(\boldsymbol{u},\boldsymbol{\varphi})$, and then, Problem (3.7) is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{X}^2_{N,\sigma}(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{X}^2_N(\Omega)$

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x = \langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega. \tag{3.8}$$

Next, we can easily check that Problem (3.1) is equivalent to Problem (3.8), and thus, Problem (3.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$.
(iii) Regularity and estimates: Let $\boldsymbol{f} \in \boldsymbol{L}^2_\sigma(\Omega)$, using the regularity result in [8, Corollary 5.4] we deduce that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ since Ω is of class $C^{2,1}$. Let us prove estimates (3.4)–(3.6). Multiplying the first equation of system (3.1) by $\overline{\boldsymbol{u}}$, integrating both sides one gets

$$\lambda\int_\Omega |\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega |\mathbf{curl}\,\boldsymbol{u}|^2\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}x.$$

Using Lemma 3.2, since $\lambda \in \Sigma_\varepsilon$, there exists a constant $C'_\varepsilon = 1/C_\varepsilon$ such that

$$\begin{aligned}|\lambda|\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} &\le C'_\varepsilon\left|\lambda\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right|\\ &\le C'_\varepsilon\left|\int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}x\right|\\ &\le C'_\varepsilon\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Thus, one has estimate (3.4). In addition, it is clear that

$$\|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \le C'_\varepsilon\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{C'^2_\varepsilon}{|\lambda|}\,\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Thus, one has estimate (3.5).

It remains to prove estimate (3.6), to this end observe that

$$\|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{f} - \lambda\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le (C'_\varepsilon + 1)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

Now, since $\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}$ estimate (3.6) follows directly. $\square$

Using Theorem 3.3, we have the following result

COROLLARY 3.4. *The operator* $-A_2$ *generates a bounded analytic semi-group on* $\boldsymbol{L}^2_\sigma(\Omega)$.

The following theorem extends Theorem 3.3 to every $1 < p < \infty$.

THEOREM 3.5. *Let* $\lambda \in \Sigma_\varepsilon$, $1 < p < \infty$ *and let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$. *The Problem (3.1) has a unique solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. *Moreover, if* $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$, *then* $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

Proof. As in the proof of Theorem 3.3, we can easily verify that Problem (3.1) is equivalent to the variational problem: Find $\boldsymbol{u} \in \boldsymbol{X}^p_{N,\sigma}(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{X}^{p'}_N(\Omega)$

$$\lambda \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}}\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x.$$

The proof is done in three steps:
(i) Case $2 \le p \le 6$. Since $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]' \hookrightarrow [\boldsymbol{H}_0^2(\mathbf{curl}, \Omega)]'$, then Problem (3.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. Next, we write Problem (3.1) in the form:

$$-\Delta \boldsymbol{u} = \boldsymbol{F}, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{3.9}$$

where $\boldsymbol{F} = \boldsymbol{f} - \lambda\,\boldsymbol{u}$. Observe that for $p \le 6$ we have:

$$\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega) \hookrightarrow \boldsymbol{L}^{p'}(\Omega) \hookrightarrow \boldsymbol{L}^{6/5}(\Omega).$$

Using the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we obtain that $\boldsymbol{u} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$, and thus, $\boldsymbol{F} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$. Next, we can verify that $\boldsymbol{F}$ satisfies the compatibility condition:

$$\forall\,\overline{\boldsymbol{v}} \in \boldsymbol{K}^{p'}_N(\Omega), \quad \langle \boldsymbol{F}, \overline{\boldsymbol{v}} \rangle_\Omega = 0,$$

where $\boldsymbol{K}^{p'}_N(\Omega)$ is defined in (2.2). Applying [8, Proposition 5.1], we deduce that the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$.
(ii) Case $p \ge 6$. Since $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]' \hookrightarrow [\boldsymbol{H}_0^{6/5}(\mathbf{curl}, \Omega)]'$, then Problem (3.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Proceeding in the same way as above, we get that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$.
(iii) Case $p \le 2$. As described above, for $p \ge 2$ the operator $\lambda\,I + A_p$ is an isomorphism from $\boldsymbol{X}^p_{N,\sigma}(\Omega)$ to $(\boldsymbol{X}^{p'}_{N,\sigma}(\Omega))'$. Then, the adjoint operator which is equal to $\lambda\,I + A'_p$ is an isomorphism from $\boldsymbol{X}^{p'}_{N,\sigma}(\Omega)$ to $(\boldsymbol{X}^p_{N,\sigma}(\Omega))'$ for $p' \le 2$. This means that the operator $\lambda\,I + A_p$ is an isomorphism for $p \le 2$, which ends the proof. Notice that the operator $\lambda\,I + A_p \in \mathcal{L}(\boldsymbol{X}^p_{N,\sigma}(\Omega), (\boldsymbol{X}^{p'}_{N,\sigma}(\Omega))')$ is defined by: for all $\boldsymbol{\varphi} \in \boldsymbol{X}^p_{N,\sigma}(\Omega)$, for all $\boldsymbol{\xi} \in \boldsymbol{X}^{p'}_{N,\sigma}(\Omega)$

$$\langle (\lambda\,I + A_p)\boldsymbol{\varphi}\,,\,\boldsymbol{\xi} \rangle_{(\boldsymbol{X}^{p'}_{N,\sigma}(\Omega))' \times \boldsymbol{X}^{p'}_{N,\sigma}(\Omega)} = \lambda \int_\Omega \boldsymbol{\varphi} \cdot \overline{\boldsymbol{\xi}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \mathbf{curl}\,\overline{\boldsymbol{\xi}}\,\mathrm{d}x.$$

Finally, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, due to [8, Corollary 5.4], the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ since Ω is of class $C^{2,1}(\Omega)$. □

Now, we want to prove a resolvent estimate similar to the estimate (3.4) for any $1 < p < \infty$. We begin with the case where the mean curvature K defined in (2.3) is positive.

PROPOSITION 3.6. *Let $\lambda \in \mathbb{C}^*$ such that $Re\,\lambda \geq 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ be the unique solution of Problem (3.1). Suppose, moreover, that the mean curvature K is positive. Then, $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_p}{|\lambda|}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.10}$$

where $C_p = \sqrt{p^2/4+1}$.

Proof. Suppose that $p \geq 2$. Thanks to Lemma 2.1, multiplying the first equation of Problem (3.1) by $|\boldsymbol{u}|^{p-2}\,\overline{\boldsymbol{u}}$ and integrating both sides we get

$$\begin{aligned}
&\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x + \int_\Omega |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2} \int_\Omega |\nabla\, |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\
&\quad + (p-2)\, i \sum_{k=1}^{3} \int_\Omega |\boldsymbol{u}|^{p-4}\, Re\left(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\right) \mathrm{Im}\left(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\right) \mathrm{d}\,x - \left\langle \frac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}}\,,\ |\boldsymbol{u}|^{p-2} \boldsymbol{u} \right\rangle_\Gamma \\
&= \int_\Omega |\boldsymbol{u}|^{p-2}\, \boldsymbol{f} \cdot \bar{\boldsymbol{u}} \,\mathrm{d}\,x,
\end{aligned} \tag{3.11}$$

where the duality on Γ is

$$\langle .\,,\ . \rangle_\Gamma = \langle .\,,\ . \rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

Since $\operatorname{div} \boldsymbol{u} = 0$ and $\boldsymbol{u} \times \boldsymbol{n} = 0$ on Γ, using (2.3) we have

$$\frac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n} = -2\, K\, \boldsymbol{u} \cdot \boldsymbol{n} \quad \text{on } \Gamma.$$

Using the fact that $\boldsymbol{u} \cdot \boldsymbol{n}$ belongs to $W^{1-1/p,p}(\Gamma)$, we can write:

$$\begin{aligned}
\left\langle \frac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}}\,,\ |\boldsymbol{u}|^{p-2} \boldsymbol{u} \right\rangle_\Gamma &= \left\langle \left(\frac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n}\right) \boldsymbol{n}\,,\ |\boldsymbol{u}|^{p-2}\, (\boldsymbol{u} \cdot \boldsymbol{n}) \boldsymbol{n} \right\rangle_\Gamma \\
&= \int_\Gamma |\boldsymbol{u}|^{p-2} \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\right) \overline{(\boldsymbol{u} \cdot \boldsymbol{n})}\, \mathrm{d}\,\sigma
\end{aligned} \tag{3.12}$$

and

$$\left(\frac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n}\right) \overline{(\boldsymbol{u} \cdot \boldsymbol{n})} = -2\, K\, |\boldsymbol{u} \cdot \boldsymbol{n}|^2 = -2\, K\, |\boldsymbol{u}|^2. \tag{3.13}$$

Now putting together (3.11)–(3.13), we get

$$
\begin{aligned}
&\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x \;+\; \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \;+\; 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\
&\quad + (p-2)\,i\,\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,Re\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\bar{\boldsymbol{u}}\Big)\mathrm{Im}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\bar{\boldsymbol{u}}\Big)\,\mathrm{d}\,x + 2\int_\Gamma K\,|\boldsymbol{u}|^p \,\mathrm{d}\,\sigma \\
&= \int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\,x.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&(Re\,\lambda)\int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x \;+\; \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \;+\; 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\
&+2\int_\Gamma K\,|\boldsymbol{u}|^p \,\mathrm{d}\,\sigma \;=\; Re\int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\,x.
\end{aligned}
\tag{3.14}
$$

Next, since $K > 0$, an easy computations shows

$$
Re\,\lambda\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\le\; \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{3.15}
$$

and

$$
\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \;\le\; \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \tag{3.16}
$$

Moreover,

$$
\begin{aligned}
&\mathrm{Im}\,\lambda\int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x \;+\; (p-2)\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,Re\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\bar{\boldsymbol{u}}\Big)\mathrm{Im}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\bar{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \\
&= \mathrm{Im}\int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\,x.
\end{aligned}
\tag{3.17}
$$

Then, putting together (3.16) and (3.17), we get

$$
|\mathrm{Im}\,\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{p}{2}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.18}
$$

Finally, putting together (3.15) and (3.18), we obtain

$$
|\lambda|^2\|\boldsymbol{u}\|^2_{\boldsymbol{L}^p(\Omega)} \;\le\; \left(\frac{p^2}{4} + 1\right)\,\|\boldsymbol{f}\|^2_{\boldsymbol{L}^p(\Omega)},
$$

which is estimate (3.10). By duality, we obtain this estimate for $1 < p < \infty$. □

Now if the mean curvature is arbitrary, then we have the following result

THEOREM 3.7. *Let $\lambda \in \mathbb{C}^*$ such that $Re\,\lambda \ge 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ be the unique solution of Problem* (3.1). *Then, $\boldsymbol{u}$ satisfies the estimates*

$$
\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\le\; \frac{\kappa_1(\Omega,\,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.19}
$$

$$
\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\le\; \frac{\kappa_2(\Omega,\,p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{3.20}
$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq \kappa_3(\Omega, p)\, \frac{1+|\lambda|}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.21}$$

Proof. **(i) Estimate (3.19).** Proceeding in the same way as in the proof of Proposition 3.6 and using (3.14), we have

$$\begin{aligned} &Re\,\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}x + \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 4\,\frac{p-2}{p^2} \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}x \\ &\quad = -2 \int_\Gamma K\, |\boldsymbol{u}|^p \,\mathrm{d}\sigma + Re \int_\Omega |\boldsymbol{u}|^{p-2}\, \boldsymbol{f} \cdot \bar{\boldsymbol{u}} \,\mathrm{d}x. \end{aligned}$$

Because Ω is of class $C^{2,1}$, the curvature K belongs to $W^{1,\infty}(\Gamma)$ and then

$$\begin{aligned} &Re\,\lambda\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 4\,\frac{p-2}{p^2} \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}x \\ &\quad \leq C_1(\Omega) \int_\Gamma |\boldsymbol{u}|^p \,\mathrm{d}\sigma + \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

In addition, using (3.17) we obtain

$$\begin{aligned} &|\lambda|\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 4\,\frac{p-2}{p^2} \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}x \\ &\quad \leq \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 2\, C_1(\Omega) \int_\Gamma |\boldsymbol{u}|^p \,\mathrm{d}\sigma + 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{3.22}$$

Next, we recall that (see [17, Chapter 1, Theorem 1.5.1.10])

$$\forall\, \varepsilon > 0, \quad \int_\Gamma |\boldsymbol{u}|^p \,\mathrm{d}\sigma \leq \varepsilon \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}x + C_\varepsilon \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}x. \tag{3.23}$$

As a result, substituting in (3.22) we obtain

$$\begin{aligned} &|\lambda|\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 4\,\frac{p-2}{p^2} \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}x \\ &\quad \leq \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 2\, C_1(\Omega) \Big[\varepsilon \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}x + C_\varepsilon \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}x \Big] \\ &\qquad + 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{3.24}$$

We chose $\varepsilon > 0$ such that $\varepsilon\, C_1(\Omega) = \frac{p-2}{p^2}$. As a result, the constant C_ε in (3.24) depends on p and Ω. Then, by setting $C_\varepsilon = C_2(\Omega, p)$ one has

$$\begin{aligned} &|\lambda| \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 2\,\frac{p-2}{p^2} \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}x \\ &\quad \leq C_3(\Omega, p)\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \end{aligned}$$

where

$$C_3(\Omega, p) = 2\,C_1(\Omega)C_2(\Omega, p). \tag{3.25}$$

We define

$$\lambda_0 = 2\,C_3(\Omega, p)$$

Step 1. Case $|\lambda| \geq \lambda_0$.

$$\frac{|\lambda|}{2}\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x$$
$$\leq \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

We distinguish three different cases of values of p:

First if $2 \leq p \leq 4$, then we can write:

$$\frac{|\lambda|}{2}\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \frac{4-p}{2}\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x$$
$$\leq 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

Thus,

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{4}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.26}$$

which is the required estimate.

Second if $p > 4$, then we write our problem in the form (3.9). Thanks to [8, Proposition 5.1], we know that

$$\|\boldsymbol{u} - \sum_{i=1}^{I}\langle\boldsymbol{u}\cdot\boldsymbol{n},\;1\rangle_{\Gamma_i}\nabla\,q_i^N\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq C_3(\Omega)\,\|\boldsymbol{f}\;-\;\lambda\,\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}.$$

Thus, using (3.26) with $p = 4$ and substituting in the last inequality, we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq \|\sum_{i=1}^{I}\langle\boldsymbol{u}\cdot\boldsymbol{n},\;1\rangle_{\Gamma_i}\nabla\,q_i^N\|_{\boldsymbol{W}^{1,4}(\Omega)} + 5\,C_3(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}. \tag{3.27}$$

Moreover, thanks to [8] we have

$$|\langle\boldsymbol{u}\cdot\boldsymbol{n}\,,\;1\rangle_{\Gamma_i}| \;\leq\; C_4(\Omega)\,\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \;\leq\; \frac{C_4(\Omega)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} \leq \frac{C_4(\Omega)}{\lambda_0}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}.$$

As a result substituting this in (3.27), we get

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq\; C_5(\Omega)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}.$$

Now, since $\boldsymbol{W}^{1,4}(\Omega) \;\hookrightarrow\; \boldsymbol{L}^\infty(\Omega)$ we get directly

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; C_6(\Omega, p)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.28}$$

Next observe that

$$\|\boldsymbol{u}\|^{p}_{\boldsymbol{L}^p(\Omega)} = \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \leq C_7(\Omega, p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

Thus, proceeding exactly as above one has

$$\begin{aligned} &Re\,\lambda\,\|\boldsymbol{u}\|^{p}_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x + 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \\ &\leq (C_3(\Omega, p)C_8(\Omega)+1)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \end{aligned}$$

and

$$|\mathrm{Im}\,\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_9(\Omega, p)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus,

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_{10}(\Omega, p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.29}$$

with some constant $C_{10}(\Omega, p) > 0$ independent of λ and $\boldsymbol{f}$, which ends the case $p > 4$. Finally, putting together (3.26) and (3.29), we obtain estimate (3.19) for all $p \geq 2$ with

$$\kappa_1(\Omega, p) = \max(4, C_{10}(\Omega, p)).$$

Third, by duality, estimate (3.19) holds for all $p < 2$.

Step 2. Case $0 < |\lambda| \leq \lambda_0$. We know that

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq C_{11}(\Omega)\left(\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right).$$

Using (3.4) and (3.5) with $\varepsilon = \pi/2$, we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C_{12}(\Omega)\frac{1+|\lambda|}{|\lambda|^2}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)},$$

Next, since $|\lambda| \leq \lambda_0$, we obtain:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_{13}(\Omega)}{|\lambda|}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)},$$

where

$$C_{13}(\Omega) = \sqrt{C_{12}(\Omega)(1+\lambda_0)}. \tag{3.30}$$

We have two different cases:

if $2 \leq p \leq 6$, then using that $\boldsymbol{H}^{\mathbf{1}}(\boldsymbol{\Omega}) \hookrightarrow \boldsymbol{L}^p(\Omega)$, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_{14}(\Omega, p)\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_{15}(\Omega, p)}{|\lambda|}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{3.31}$$

Next if $p \geq 6$, then proceeding exactly in the same way as in Step 1 (case $p > 4$) we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_{16}(\Omega, p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.32}$$

(ii) Estimate (3.20). Let $\boldsymbol{u} \in \mathbf{D}(A_p)$ be the unique solution of Problem (3.1) and set

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \sum_{i=1}^{I} \langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\ 1 \rangle_{\Gamma_i} \nabla q_i^N .$$

Observe that $\langle \widetilde{\boldsymbol{u}} \cdot \boldsymbol{n}\,,\ 1 \rangle_{\Gamma_i} = 0$, $1 \leq i \leq I$. Thus, using [8, Corollary 5.4] we know that $\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\Delta \widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)}$. Using the Gagliardo–Nirenberg inequality (cf. [1, Chapter IV, Theorem 4.14, Theorem 4.17]), we have

$$\|\mathbf{curl}\, \widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, \|\Delta \widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Then

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \|\mathbf{curl}\, \widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\Delta \widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} = \|\Delta \boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Moreover, thanks to [8] we know that

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

As a consequence,

$$\begin{aligned}
\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &\leq C(\Omega, p)\, \|\Delta \boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \\
&= C(\Omega, p)\, \|\boldsymbol{f} - \lambda\, \boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \\
&\leq \frac{C(\Omega, p)}{\sqrt{|\lambda|}}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},
\end{aligned}$$

and thus, we deduce estimate (3.20).
(iii) Estimate (3.21). Since Ω is of class $C^{2,1}$, on the domain $\mathbf{D}(A_p)$ the norm of $\boldsymbol{W}^{2,p}(\Omega)$ is equivalent to the graph norm of the Stokes operator. As a result, estimate (3.21) holds. □

The above results allow us to deduce the following:

THEOREM 3.8. *The operator $-A_p$ defined in the previous section generates a bounded analytic semi-group on $\boldsymbol{L}^p_\sigma(\Omega)$ for all $1 < p < \infty$.*

REMARK 3.9. Köhne considered in [19, Chapter 3] the Stokes operator subject to the pressure boundary condition (1.2) in a bounded domain Ω with a boundary Γ of class C^3. He proved [19, Proposition 3.40, Proposition 3.42] that for all $1 < p < \infty$ with $p \neq \frac{3}{2}$ and $p \neq 3$, and under some compatibility conditions, the Stokes operator with pressure boundary conditions is a closed densely defined operator. He also proved [19, Corollary 3.44] using L^p theory of abstract parabolic evolution equation that under the previous assumptions the Stokes operator with pressure boundary conditions has the property of maximal L^p regularity on finite time interval. Köhne deduced from this maximal L^p regularity that the Stokes operator is sectorial and generates a bounded analytic semi-group.

The analyticity on $\boldsymbol{L}^p_\sigma(\Omega)$ allows us to obtain strong solutions to the Problem (1.1)–(1.2). In order to obtain weak solutions, we shall study the resolvent of the Stokes operator on $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\ \Omega)]'_\sigma$.

THEOREM 3.10. *Let* $\lambda \in \mathbb{C}^*$ *such that* $Re\,\lambda \geq 0$ *and let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'$ *with* $1 < p < \infty$. *The Problem* (1.3) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *satisfying*

$$\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} \tag{3.33}$$

for some constant $C(\Omega, p) > 0$ *independent of* λ *and* $\boldsymbol{f}$.

Proof. (i) For the existence of solutions for Problem (1.3), we proceed in the same way as in Theorem 3.3, Theorem 3.5 and the proof of [8, Theorem 4.10].
(ii) To prove estimate (3.33), we proceed as follows: Consider the problem:

$$\begin{cases} \lambda \boldsymbol{v} - \Delta \boldsymbol{v} - \nabla\theta = \boldsymbol{F}, & \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}, & \theta = 0 \quad \text{on } \Gamma, \end{cases} \tag{3.34}$$

where $\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)$ and $\lambda \in \mathbb{C}^*$ such that $Re\,\lambda \geq 0$. It is clear that θ is a solution of following Dirichlet problem:

$$-\Delta\theta = \operatorname{div}\boldsymbol{F} \quad \text{in } \Omega \quad \text{and} \quad \theta = 0 \quad \text{on } \Gamma.$$

Then, θ belongs to $W_0^{1,p'}(\Omega)$ and satisfies the estimate

$$\|\theta\|_{W^{1,p'}(\Omega)} \leq C(\Omega, p')\, \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Moreover, by moving $\nabla\theta$ in problem (3.34) to the right-hand side and using Theorem 3.7, we deduce that the Problem (3.34) has a unique solution $\boldsymbol{v} \in \boldsymbol{W}^{2,p'}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}. \tag{3.35}$$

Applying the curl operator to the first equation of (3.34), we obtain that $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{v}$ is solution of the following problem:

$$\begin{cases} \lambda \boldsymbol{z} - \Delta \boldsymbol{z} = \mathbf{curl}\,\boldsymbol{F}, & \operatorname{div} \boldsymbol{z} = 0 \quad \text{in } \Omega, \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma. \end{cases}$$

Using the result of [4, Theorems 4.10-4.11], we deduce that $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{2,p'}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \|\mathbf{curl}\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}. \tag{3.36}$$

Combining (3.35) and (3.36), we deduce that

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}.$$

Now let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ be the solution of Problem (1.3), then

$$\begin{aligned}
\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} &= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega),\, \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\, \boldsymbol{F}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}}\\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega),\, \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\, \lambda\, \boldsymbol{v} - \Delta \boldsymbol{v} - \nabla\theta\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}}\\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega),\, \boldsymbol{F}\neq 0} \frac{|\langle \lambda\, \boldsymbol{u} - \Delta \boldsymbol{u} - \nabla\pi\,,\, \boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}}\\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega),\, \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{f}\,,\, \boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}}\\
&\leq \frac{C(\Omega, p')}{|\lambda|}\, \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'},
\end{aligned}$$

which is estimate (3.33). □

Now we define the operator B_p as the extension of the Stokes operator A_p to the space $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$ by:

$$B_p \;:\; \mathbf{D}(B_p) \subset [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma \longmapsto [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$$

and

$$\forall\, \boldsymbol{u} \in \mathbf{D}(B_p), \quad B_p\, \boldsymbol{u} = -\Delta \boldsymbol{u} \quad \text{in } \Omega. \tag{3.37}$$

The domain of B_p is given by

$$\begin{aligned}
\mathbf{D}(B_p) = \Big\{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta\boldsymbol{u} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]',\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\\
\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\Big\}.
\end{aligned} \tag{3.38}$$

The domain $\mathbf{D}(B_p)$ is dense in $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$, and the proof is done in two steps.

PROPOSITION 3.11. *The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$.*

Proof. Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$, $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\boldsymbol{\xi}$. Using the density of $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ in $\boldsymbol{L}^p_\sigma(\Omega)$ (see [7]) and the fact that $\operatorname{div} \boldsymbol{f} = \operatorname{div} \boldsymbol{\psi} = 0$ in Ω, we know that there exists a sequence $(\boldsymbol{\psi_k})_{k\in\mathbb{N}}$ in $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ such that the sequence $(\boldsymbol{\psi}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{\psi}$ in $\boldsymbol{L}^p(\Omega)$. On the other hand, there exists a sequence $(\boldsymbol{\xi_k})_{k\in\mathbb{N}}$ in $\boldsymbol{\mathcal{D}}(\Omega)$ such that $(\boldsymbol{\xi}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{\xi}$ in $\boldsymbol{L}^p(\Omega)$. Thus, the sequence $(\mathbf{curl}\, \boldsymbol{\xi}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{curl}\, \boldsymbol{\xi}$ in $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'$. Now, for all $k \in \mathbb{N}$ we set $\boldsymbol{f}_k = \boldsymbol{\psi}_k + \mathbf{curl}\, \boldsymbol{\xi}_k$. It is clear that $(\boldsymbol{f}_k)_{k\in\mathbb{N}}$ is a sequence in $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ and $(\boldsymbol{f}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{f}$ in $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'$. □

COROLLARY 3.12. *The domain $\mathbf{D}(B_p)$ of the operator B_p defined by (3.38) is dense in $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$.*

Proof. Observe that $\mathbf{D}(A_p) \subset \mathbf{D}(B_p) \subset \boldsymbol{L}^p_\sigma(\Omega)$, where $\mathbf{D}(A_p)$ is given by (2.4). The desired result is obtained using the density of $\mathbf{D}(A_p)$ in $\boldsymbol{L}^p_\sigma(\Omega)$ which is proved in Proposition 2.2 and using Proposition 3.11 which gives us that $\boldsymbol{L}^p_\sigma(\Omega)$ is dense in $[\boldsymbol{H}^{p'}_0(\mathbf{curl}, \Omega)]'_\sigma$. □

Notice that Theorem 3.10 allows us to deduce the following theorem.

THEOREM 3.13. *The operator $-B_p$ generates a bounded analytic semi-group on $[\boldsymbol{H}^{p'}_0(\mathbf{curl}, \Omega)]'_\sigma$ for all $1 < p < \infty$.*

REMARK 3.14. **(i)** We can consider the Stokes problem with adding the flux through the connected components Γ_i:

$$\langle \boldsymbol{u} \cdot \boldsymbol{n},\ 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \tag{3.39}$$

The associated Stokes operator A'_p which is the restriction of A_p to the space $\boldsymbol{X}_p$ defined by

$$\boldsymbol{X}_p = \{\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega);\ \forall\, \boldsymbol{v} \in \boldsymbol{K}^{p'}_N(\Omega),\ \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x = 0\}, \tag{3.40}$$

is a well-defined operator with dense domain

$$\mathbf{D}(A'_p) = \left\{\boldsymbol{u} \in \mathbf{D}(A_p);\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\ 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I\right\}$$

and generates a bounded analytic semi-group on $\boldsymbol{X}_p$ for $1 < p < \infty$. In addition, thanks to [8] we know that the operator A'_p is invertible with bounded inverse (i.e., $0 \in \rho(A'_p)$), this is very important in the sequel and gives us an exponential decay of the solution of the Stokes Problem (1.4) when $\boldsymbol{u}_0 \in \boldsymbol{X}_p$ and $\boldsymbol{f} = \mathbf{0}$. Notice that, obviously, A_p coincides with A'_p in the case where Γ is connected.
ii) We can also consider the space

$$\boldsymbol{Y}_p = \left\{\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathbf{curl}, \Omega)]'_\sigma;\ \forall\, \boldsymbol{v} \in \boldsymbol{K}^{p'}_N(\Omega),\ \langle \boldsymbol{f},\ \boldsymbol{v}\rangle_\Omega = 0\right\}.$$

We can also verify that the operator B'_p which is the restriction of B_p to the space $\boldsymbol{Y}_p$ is a well-defined operator of dense domain

$$\mathbf{D}(B'_p) = \left\{\boldsymbol{u} \in \mathbf{D}(B_p);\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\ 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I\right\},$$

and generates a bounded analytic semi-group on $\boldsymbol{Y}_p$ for $1 < p < \infty$.

4. The time-dependent homogeneous Stokes problem

In this section, we solve the time-dependent homogeneous Stokes Problem (1.1) with the boundary condition (1.2) using the semi-group theory. As described in (2.5), due to the boundary conditions (1.2) the Stokes operator coincides with the $-\Delta$ operator with the corresponding boundary conditions and our work is reduced to study the Problem (1.4).

4.1. Weak solutions

As in the case of the heat equation, the following theorem shows that the solution of Problem (1.4) with $\boldsymbol{f} = \boldsymbol{0}$ is regular for $t > 0$. Moreover, it allows us to describe the decay in time of this solutions. The proof of the following theorem is a direct application of the analyticity of the Stokes operator with pressure boundary conditions proved in the previous section.

THEOREM 4.1. *Let $1 < p < \infty$ and suppose that $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ and $\boldsymbol{f} = \boldsymbol{0}$. Then, Problem* (1.4) *has a unique solution $\boldsymbol{u}(t)$ satisfying*

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{L}^p_\sigma(\Omega)) \cap C(]0, +\infty[, \mathbf{D}(A_p)) \cap C^1(]0, +\infty[, \boldsymbol{L}^p_\sigma(\Omega)),$$
$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(A_p^\ell)), \quad \forall k \in \mathbb{N}, \ \forall \ell \in \mathbb{N}^*.$$

Moreover, we have the estimates

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{4.1}$$

and

$$\Big\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \Big\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

In addition, if Ω is of class $C^{2,1}$ the following estimates hold

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{\sqrt{t}}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{4.2}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega, p)\, (1 + \frac{1}{t})\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.3}$$

In the following theorem, we consider a particular case, where the initial data $\boldsymbol{u}_0$ belongs to $\boldsymbol{X}_p$ [see (3.40)]. This leads to an exponential decay of the solution $\boldsymbol{u}$ with respect to time. As described in Remark 3.14, on the space $\boldsymbol{X}_p$ the Stokes operator with normal and pressure boundary conditions is invertible with bounded inverse and generates a bounded analytic semi-group. Thus, using the result of Pazy [24, Chapter 4, Theorem 4.3] we obtain the following theorem.

THEOREM 4.2. *Suppose that $\boldsymbol{u}_0 \in \boldsymbol{X}_p$ with $1 < p < \infty$ and $\boldsymbol{f} = \boldsymbol{0}$. Then, Problem* (1.4) *has a unique solution $\boldsymbol{u}$ satisfying*

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{X}_p) \cap C(]0, +\infty[, \mathbf{D}(A'_p)) \cap C^1(]0, +\infty[, \boldsymbol{X}_p),$$
$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(A'^{\ell}_p)), \quad \forall k, \ \ell \in \mathbb{N}.$$

Moreover, there exist constants $M, \mu > 0$ such that the solution $\boldsymbol{u}$ satisfies the estimates:

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq M\, C(\Omega, p)\, e^{-\mu t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$\left\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|_{\boldsymbol{L}^p(\Omega)} \leq M\, C(\Omega,\, p)\, \frac{e^{-\mu\, t}}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}$$

In addition, if Ω *is of class* $C^{2,1}$ *the following estimates hold*

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq M\, C(\Omega,\, p)\, \frac{e^{-\mu\, t}}{\sqrt{t}}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega,\, p)\, \frac{e^{-\mu\, t}}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

The following theorem gives weak solution for problem (1.4).

THEOREM 4.3. *Let* $1 < p < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ *and let* $\boldsymbol{f} = \boldsymbol{0}$. *Then, the unique solution* $\boldsymbol{u}$ *of Problem* (1.4) *satisfies:*

$$\boldsymbol{u} \in L^q(0, T_0;\; \boldsymbol{W}^{1,p}(\Omega)), \quad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\; [\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'_\sigma), \tag{4.4}$$

with $1 < q < 2$ *and* $0 < T \leq \infty$ *and some constant* C *independent of* T.

Proof. Thanks to [8] we know that

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \simeq \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}.$$

Since $\boldsymbol{u}$ satisfies (4.1)–(4.2), the solution $\boldsymbol{u}$ clearly belongs to $\in L^q(0, T_0;\; \boldsymbol{W}^{1,p}(\Omega))$ for all $1 < q < 2$ and $0 < T < \infty$. To prove that $\frac{\partial \boldsymbol{u}}{\partial t} = \Delta \boldsymbol{u} \in L^q(0, T;\; [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]')$, we set

$$\widetilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - \sum_{i=1}^{I} \langle \boldsymbol{u}(t) \cdot \boldsymbol{n}\,,\; 1\rangle_{\Gamma_i} \mathbf{grad}\, q_i^N.$$

Applying [8, Corollary 4.4], we have

$$\|\Delta \boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} = \|\Delta \widetilde{\boldsymbol{u}}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} \simeq \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)},$$

this end the proof. □

4.2. Strong solutions

THEOREM 4.4. *Suppose that* $1 < p < \infty$ *and* $\boldsymbol{u}_0 \in \boldsymbol{X}^p_{N,\sigma}(\Omega)$ *(given by* (2.1)*) and* $\boldsymbol{f} = \boldsymbol{0}$. *The unique solution* $\boldsymbol{u}(t)$ *of Problem* (1.4) *satisfies in particular*

$$\boldsymbol{u} \in L^q(0, T;\; \boldsymbol{W}^{2,p}(\Omega)) \quad \text{and} \quad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\; \boldsymbol{L}^p_\sigma(\Omega)), \tag{4.5}$$

with $1 \leq q < 2$ *and* $T < \infty$. *Moreover, the following estimate holds*

$$\left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C}{\sqrt{t}} \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{4.6}$$

Proof. Let $\boldsymbol{u}_0 \in \boldsymbol{X}^p_{\sigma,N}(\Omega)$ and let $\boldsymbol{u}(t)$ be the unique solution of Problem (1.4). Set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}(t)$, we can verify that $\boldsymbol{z}(t)$ is a solution of the problem

$$\begin{cases} \frac{\partial \boldsymbol{z}}{\partial t} - \Delta \boldsymbol{z} = \boldsymbol{0}, \ \operatorname{div} \boldsymbol{z} = 0 & \text{in } \ \Omega \times (0, T), \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0, \qquad \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma \times (0, T), \\ \qquad\qquad \boldsymbol{z}(0) = \mathbf{curl}\, \boldsymbol{u}_0 & \text{in } \ \Omega. \end{cases} \tag{4.7}$$

Thanks to [3] we know that the Stokes operator associated with Problem (4.7) generates a bounded analytic semigroup on the space of divergence free functions whose normal components vanishes on Γ and that

$$\|\mathbf{curl}\, \boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{C(\Omega, p)}{\sqrt{t}} \|\boldsymbol{z}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

This means that

$$\left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|_{\boldsymbol{L}^p(\Omega)} = \|\Delta \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{C(\Omega, p)}{\sqrt{t}} \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

Thus, $\frac{\partial \boldsymbol{u}}{\partial t}$ satisfies estimate (4.6) and $\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T; \boldsymbol{L}^p_\sigma(\Omega))$ for all $1 \le q < 2$ and for all $T < \infty$. Finally using that $\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta \boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}$, (4.5) follows directly. $\square$

Appendix

In this appendix, we prove the existence of a trace operator for a subspace of $W^{-1,p}(\Omega)$ to be determined. This result was proved in [20, Theorem 6.5, Chapter 2] for $p = 2$.

To this end, we introduce the space:

$$M_p(\Omega) = \left\{ v \in W^{-1,p}(\Omega); \ \ \Delta v \in W^{-2+1/p,p}(\Omega) \right\},$$

which is a reflexive Banach space for the norm

$$\|v\|_{M_p(\Omega)} = \|v\|_{W^{-1,p}(\Omega)} + \|\Delta v\|_{W^{-2+1/p,p}(\Omega)}.$$

LEMMA 4.5. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{2,1}$. The space $\mathcal{D}(\overline{\Omega})$ is dense in $M_p(\Omega)$.*

Proof. Let ℓ be a continuous linear form in $M_p(\Omega)$ (i.e., $\ell \in \left(M_p(\Omega)\right)'$). Then, there exists a pair $(f, g) \in W^{1,p'}_0(\Omega) \times W^{2-1/p,p'}_0(\Omega)$, such that for every function v in $M_p(\Omega)$ we have

$$\langle \ell, v \rangle = \langle f \,, \ v \rangle_{W^{1,p'}_0(\Omega) \times W^{-1,p}(\Omega)} + \langle \Delta v, g \rangle_{W^{-2+1/p,p}(\Omega) \times W^{2-1/p,p'}_0(\Omega)}. \tag{4.8}$$

Thanks to the Hahn–Banach theorem, it suffices to show that any ℓ which vanishes on $\mathcal{D}(\overline{\Omega})$ is actually zero on $M_p(\Omega)$. Let us suppose that $\ell = 0$ on $\mathcal{D}(\overline{\Omega})$, thus on $\mathcal{D}(\Omega)$. Then, we can deduce from (4.8) that

$$f + \Delta g = 0 \quad \text{in } \Omega,$$

and hence, we have $\Delta g \in W_0^{1,p'}(\Omega)$, and then $g \in W^{3,p'}(\Omega) \cap W_0^{2-1/p,p'}(\Omega)$. Let $\tilde{f} \in W^{1,p'}(\mathbb{R}^N)$ and $\tilde{g} \in W^{2-\frac{1}{p},\,p'}(\mathbb{R}^N)$ be, respectively, the extensions by 0 of f and g to $\mathbb{R}^N$. From (4.8), we get $\tilde{f} + \Delta\tilde{g} = 0$ in $\mathbb{R}^N$, and thus $\Delta\tilde{g} \in W^{1,p'}(\mathbb{R}^N)$. Now, according to the properties for Δ in $\mathbb{R}^N$ (see [6]), we can deduce that $\tilde{g} \in W^{3,\,p'}(\mathbb{R}^N)$. Since $\tilde{g}$ is an extension by 0, it follows that $g \in W_0^{3,\,p'}(\Omega)$. Then, by density of $\mathcal{D}(\Omega)$ in $W_0^{3,\,p'}(\Omega)$, there exists a sequence $(\varphi_k)_k \subset \mathcal{D}(\Omega)$ such that $\varphi_k \to g$ in $W^{3,\,p'}(\Omega)$. Thus, for any $v \in M_p(\Omega)$, we have

$$\begin{aligned}
\langle \ell, v\rangle &= -\int_\Omega v \cdot \Delta\overline{g}\,\mathrm{d}x + \langle \Delta v, g\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,p'}(\Omega)}\\
&= \lim_{k\to\infty}\Big(-\int_\Omega v\cdot\Delta\overline{\varphi_k}\,\mathrm{d}x + \langle\Delta v, \varphi_k\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,p'}(\Omega)}\Big)\\
&= 0,
\end{aligned}$$

i.e. ℓ is identically zero. □

To study the traces of functions which belong to $M_p(\Omega)$, we have the following lemma.

LEMMA 4.6. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{2,1}$. The linear mapping $\gamma_0 : v \longmapsto v|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$\gamma_0 : M_p(\Omega) \longrightarrow W^{-1-1/p,\,p}(\Gamma).$$

Moreover, we have the Green formula:

$$\forall v \in M_p(\Omega), \quad \forall\varphi \in W^{3,\,p'}(\Omega)\cap W_0^{1,\,p'}(\Omega),$$

$$\begin{aligned}
&\int_\Omega v\Delta\overline{\varphi}\,\mathrm{d}x - \langle\Delta v,\varphi\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,p'}(\Omega)}\\
&\qquad = \left\langle v, \frac{\partial\varphi}{\partial\boldsymbol{n}}\right\rangle_{W^{-2+1/p',\,p}(\Gamma)\times W^{2-1/p',\,p'}(\Gamma)}.
\end{aligned} \tag{4.9}$$

Proof. Let $v \in \mathcal{D}(\overline{\Omega})$ and $\varphi \in W^{3,\,p'}(\Omega)\cap W_0^{1,\,p'}(\Omega)$, then formula (4.9) obviously holds. Next observe that, for every $\mu \in W^{1+1/p,\,p'}(\Gamma)$, there exists a function $\varphi \in W^{3,\,p'}(\Omega)\cap W_0^{1,\,p'}(\Omega)$ such that $\frac{\partial\varphi}{\partial\boldsymbol{n}} = \mu$ on Γ and $\frac{\partial^2\varphi}{\partial\boldsymbol{n}^2} = 0$ on Γ. Moreover, we have

$$\|\varphi\|_{W^{3,\,p'}(\Omega)} \le C\,\|\mu\|_{W^{1+1/p,\,p'}(\Gamma)}.$$

Consequently,

$$\left|\,\langle v,\mu\rangle_{W^{-1-1/p,\,p}(\Gamma)\times W^{1+1/p,\,p'}(\Gamma)}\,\right| \le C\,\|v\|_{M_p(\Omega)}\,\|\mu\|_{W^{1+1/p,\,p'}(\Gamma)}.$$

Thus,

$$\|v\|_{W^{-1-1/p,\,p}(\Gamma)} \leq C\ \|v\|_{M_p(\Omega)}.$$

We can deduce that the linear mapping γ is continuous for the norm of $M_p(\Omega)$. Since $\mathcal{D}(\overline{\Omega})$ is dense in $M_p(\Omega)$, γ can be extended by continuity to $\gamma \in \mathcal{L}\left(M_p(\Omega); W^{-1-1/p,\,p}(\Gamma)\right)$ and formula (4.9) holds for all $v \in M_p(\Omega)$ and $\varphi \in W^{3,\,p'}(\Omega) \cap W_0^{1,\,p'}(\Omega)$. $\square$

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