Discontinuous Galerkin method for the incompressible Magnetohydrodynamic system with Navier-type boundary condition

Julien Poirier*, Nour Seloula†

Résumé

In this work, we introduce and analyze a discontinuous Galerkin method (DG) for the stationary Magnetohydrodynamic system (MHD) with Navier-type boundary condition for both the velocity and the magnetic field. We prove a new discrete Sobolev inequality for the L^p-norm which is the key ingredient in the study of the well-posedness and the convergence of the DG scheme. The existence of the discrete solution is proved by using Brouwer's theorem under assumption of sufficiently small data. We provide *a priori* error estimates in terms of a natural energy norm for the velocity and the magnetic fields. The novelty of this work is that, to the author's knowledge, this is the first time that a DG method, applied to the nonlinear coupled MHD system, with Navier-type boundary conditions for both the velocity and the magnetic field, is proposed and completely analyzed.

Key words : Magnetohydrodynamic ; discontinuous Galerkin method ; fixed point theory ; Navier boundary condition ; L^p discrete Sobolev inequality, *a priori* error estimates.

Mathematics subject classifications (2000) : 65N12, 65N15, 65N22 65N30, 35Q30, 35Q60.

1 Introduction

Magnetohydrodynamics (abbreviated as "MHD") is the field of physics that describes the behavior of an electrically conductive fluid (such as liquid metals, plasmas, saltwater, etc.) under the influence of magnetic fields [14,17]. This area of physics was discovered in 1942 by H. Alfvén in [1] (for which he received the Nobel Prize in 1942), following the work of Hartmann on liquid metals in 1937. MHD plays a predominant role in the field of astrophysics. For example, one can mention the phenomenon of solar winds, which is the ejection of plasma from the sun [39]. Also, MHD has been highlighted in recent years with experiments on nuclear fusion, particularly with tokamaks (see for example [41]). The MHD is also found in various fields of industry and engineering, for example for cooling nuclear reactors with liquid metal, pumping metals using electromagnetic pumps, simulating electrolysis during aluminum production, MHD energy production (see [17,26,29]).

MHD is characterized by a system of coupled partial differential equations : fluid mechanics is governed by the Navier-Stokes equations, while electromagnetism is described by Maxwell's equations.

We consider the particular case of stationary incompressible MHD, meaning that the viscous fluids are incompressible. We thus obtain the following system of partial differential equations (see,

*Normandie Univ., UNICAEN, CNRS, Laboratoire de Mathématiques Nicolas Oresme, UMR CNRS 6139, 14000 Caen, France. E-mail : `julien.poirier.prof@gmail.com`.

†Normandie Univ., UNICAEN, CNRS, Laboratoire de Mathématiques Nicolas Oresme, UMR CNRS 6139, 14000 Caen, France. E-mail : `nour-elhouda.seloula@unicaen.fr`.

e.g., [14, 17, 20]) :

$$-\nu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \nabla p - \kappa(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} = \boldsymbol{f} \qquad \text{in}\,\Omega \tag{1.1a}$$

$$\kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{g} \qquad \text{in}\,\Omega \tag{1.1b}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \qquad \text{in}\,\Omega. \tag{1.1c}$$

Here Ω is a bounded simply connected domain in $\mathbb{R}^3$ with a connected boundary Γ. Additional smoothness of the boundary will be precised whenever needed. The unknowns of these equations are the velocity field $\boldsymbol{u}$, the magnetic field $\boldsymbol{b}$, and the pressure p, while $\boldsymbol{f}$ represents external forces and $\boldsymbol{g}$ is a source term. The parameters ν, μ, and κ are respectively the fluid viscosity, magnetic permeability, and coupling term. The term $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ represents the Lorentz force and the term $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ represents the motion of the conducting material in the previously mentioned magnetic field.

Now, we complete the MHD system (1.1) with boundary conditions. For the magnetic field, we consider the following boundary condition

$$\boldsymbol{b} \cdot \boldsymbol{n} = 0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \quad \text{on}\,\Gamma, \tag{1.2}$$

where the vector $\boldsymbol{n}$ stands for the outward unit normal on Γ. For the velocity field, the boundary conditions most frequently considered in the study of the MHD problem are Dirichlet conditions : $\boldsymbol{u} = \mathbf{0}$ on Γ which are the no-slip conditions, meaning that the fluid adheres to the wall. However, this formulation, introduced by G. Stokes in 1845 [38], has limitations. Indeed, according to the work of Serrin [37], these conditions may not be realistic, leading to phenomena of boundary layer. To model different situations, several types of boundary conditions must be considered. In this paper, we will consider the following boundary condition known as Navier-type boundary condition :

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on}\,\Gamma, \tag{1.3}$$

The scalar function p will be required to have zero mean over Ω. The data $\boldsymbol{g}$ should satisfy the following compatibility condition

$$\operatorname{div}\boldsymbol{g} = 0 \quad \text{in}\,\Omega \qquad \text{and} \quad \boldsymbol{g} \cdot \boldsymbol{n} = 0 \ \ \text{on}\,\Gamma. \tag{1.4}$$

There is significant literature devoted to the study of numerical schemes for solving the MHD equations with Dirichlet boundary conditions for the velocity field. Several discretizations based on conforming finite elements have been proposed for both the linear and nonlinear cases of the MHD system [6, 16, 20]. In the case of a non-convex polyhedral domain Ω, a conforming finite element method with regularization is analyzed in [21]. This method allows for the enforcement of the constraint $\nabla \cdot \boldsymbol{b} = 0$ in the discrete formulation. Another way to address the difficulties related to the approximation of the magnetic field has been proposed in [34, 35] by using Nédélec finite elements and introducing a Lagrange multiplier. The MHD system has also been approached by the discontinuous Galerkin (DG) method. The first result concerning the approximation of the linearized MHD system, with Dirichlet boundary conditions for the velocity field, by the DG method is found in [25]. In the recent work [32], a DG method is proposed and analyzed for the nonlinear MHD problem with two types of boundary conditions for the magnetic field but still with Dirichlet type boundary conditions for the velocity field. In this paper, we consider a new variational setting for the formulation of MHD problem where the Navier-type boundary condition is imposed both for the velocity and for the magnetic field. To our knowledge, there is no work in the literature done concerning the DG method for the MHD problem (1.1) with boundary conditions (1.2)-(1.3).

DG method is a class of finite element methods that use completely discontinuous basis functions. These basis and test functions are chosen from the same space without any continuity imposed at the interfaces between elements (a tetrahedron or hexahedron in 3D, a triangle or quadrilateral in 2D). This specificity allows this method great flexibility in meshing. Indeed, in the absence of sensitivity to mesh regularity, the method is suitable for representing industrial geometries that are sometimes complex and require non-structured and non-conforming meshes. In addition, the approximation with this method requires the use of a local weak formulation where the boundary conditions are taken into account weakly, directly in the formulation of the problem without imposing them in the definition of the approximation space. An integration by parts then reveals boundary terms where the physical fluxes are approximated by numerical fluxes at the interfaces due to the discontinuities between the meshes. This method has other important properties. It is locally conservative, easily parallelizable, giving the method great interest. Introduced exactly 48 years ago by Reed and Hill (1973) [33] for the resolution of neutron transport equations, the DG method has undergone a number of developments in recent years and is still a widely used method today. Its field of application has subsequently been extended to problems with diffusion, including [7, 11, 19] for Navier-Stokes equations and [13, 22, 28] for electromagnetism.

The main difficulty in approximating Maxwell's equations is incorporating the constraint $\operatorname{div} \boldsymbol{b} = 0$ into the DG scheme. This constraint can be imposed by the so-called regularization technique. This method formally consists of replacing the operator $\mathbf{curl}(\mathbf{curl})$ with the vector Laplacian operator by adding a term $\mathbf{grad}(\operatorname{div})$ to the equation $\kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b}) = \boldsymbol{g}$ in system (1.1b). This term is zero because every solution $\boldsymbol{b}$ of problem (1.1b) has zero divergence. Thus, according to the identity :

$$\mathbf{curl}(\mathbf{curl}) - \mathbf{grad}(\operatorname{div}) = -\Delta,$$

every magnetic field $\boldsymbol{b}$ that solves (1.1b) also satisfies the equation

$$-\kappa\mu\Delta\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b}) = \boldsymbol{g} \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega.$$

This method then allows for a naturally regularized problem posed in the space

$$\boldsymbol{H}_0(\operatorname{div},\Omega)\cap\boldsymbol{H}(\mathbf{curl},\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega);\quad \operatorname{div}\boldsymbol{v}\in L^2(\Omega),\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^2(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma\}.$$

If the domain Ω is polyhedral and convex or has a regular boundary (c.f. [18, Theorem 3.7] and [5, Theorem 3.4]), the space $\boldsymbol{H}_0(\operatorname{div},\Omega)\cap\boldsymbol{H}(\mathbf{curl},\Omega)$ coincides with the space

$$\boldsymbol{H}^1_T(\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma\}. \tag{1.5}$$

The regularization technique for Maxwell's equations was initially introduced by Werner [40] and Leis [27]. Its applications are diverse, including the study of Maxwell's equations in harmonic and low frequency regimes [30], the study of Maxwell's equations in a polyhedral domain [12], and the study of a domain decomposition method for Maxwell's equations [2]. However, when the domain has geometric singularities, such as corners or re-entrant edges, the approach is not suitable, and the DG solution of the regularized problem may converge but not to the solution of the original physical problem (see the numerical results in [23]). The same situation was observed in the case of low-frequency approximation by conforming finite elements (see [2, 12]). This is due to the lack of regularity of the solution which, in this case, is not in $\boldsymbol{H}^1(\Omega)$ but only in $\boldsymbol{H}^{1/2}(\Omega)$ [18]. In this case, an alternative approach consists of introducing a Lagrange multiplier r as an additional unknown of the problem [25, 32]. This Lagrange multiplier r can be seen as a pseudo-pressure associated with the constraint of zero divergence satisfied by any solution of the problem. The size of the discrete system then becomes larger and, instead of the second equation of system (1.1b), the couple $(\boldsymbol{b}, r)$ is a solution of :

$$\kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} + \nabla r - \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d}) = \boldsymbol{g} \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \qquad \text{in } \Omega,$$

with a boundary condition $r = 0$ on Γ. Note here that by taking the divergence of the previous equation, we obtain $\Delta r = \operatorname{div}\boldsymbol{g}$ in Ω and in the particular case where the source term $\boldsymbol{g}$ has zero divergence, we have $r = 0$.

This method was applied in [25] for the approximation of the linearized MHD system. It has recently been extended to the nonlinear case in [32]. However, it should be emphasized that the boundary condition considered in these works for the velocity field is the classical Dirichlet-type condition. In our work, we propose a regularized or augmented DG formulation to impose the two constraints $\operatorname{div} \boldsymbol{u} = 0$ and $\operatorname{div} \boldsymbol{b} = 0$.

The main difficulty is to prove the stability of the bilinear forms arising from the Navier-Stokes convection terms O_h and the coupling term C_h on the discrete spaces. In order to pay particular attention to the stability of the velocity/magnetic field coupling form C_h, let us clarify its definition here :

$$C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{b}_h) := \kappa \sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{v}_h \times \boldsymbol{b}_h) \cdot \mathbf{curl}\, \boldsymbol{b}_h \, d\boldsymbol{x} - \kappa \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{v}_h \times \boldsymbol{b}_h\}\!\} \cdot [\![\boldsymbol{b}_h]\!]_T \, ds. \tag{1.6}$$

In a recent work [32], the authors propose an analysis of a DG scheme for the (MHD) system in $\mathbb{R}^3$ where the boundary conditions are of Dirichlet type for the velocity field and Navier type (1.2) for the magnetic field. They consider the general case of a Lipschitz domain Ω. They show the continuity of C_h by applying a L^6-L^3-L^2 argument for the given term on the elements, then the discrete trace inequality is used for the given term on the boundary. This approach is natural in the case of a non-smooth domain. Indeed, without any regularity assumption on the domain, and given the boundary conditions considered for the magnetic field $\boldsymbol{b}$, the solution $\boldsymbol{b}$ of the system (2.1) is only in $\boldsymbol{H}^{1/2}(\Omega)$ and hence in $\boldsymbol{L}^3(\Omega)$ by Sobolev embedding. However, the solution $\boldsymbol{u}$ belongs to $\boldsymbol{H}^1(\Omega)$ and hence to $\boldsymbol{L}^6(\Omega)$, which allows us to apply the discrete inequality derived in [19, 19] (see also [15]), which states that the $\boldsymbol{L}^6$ norm is controlled by the following DG norm denoted by $|\!|\!|\cdot|\!|\!|_h$ (see (2.19) with $r = 6$) :

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^6(\Omega)} \leq C|\!|\!|\boldsymbol{v}_h|\!|\!|_h := C\Big(\sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{h_e} \|[\![\boldsymbol{v}_h]\!]\|_{0,e}^2\Big)^{1/2}. \tag{1.7}$$

Since the DG norm for $\boldsymbol{b}$ is different (see the definition of $|\boldsymbol{b}_h|_C$ in (1.8)), a discrete functional analysis is established in [32] to show that the $\boldsymbol{L}^3$ norm is controlled by this DG norm. More precisely, the following discrete inequality is shown in [32] for the piecewise polynomial space :

$$\|\boldsymbol{b}_h\|_{\boldsymbol{L}^3(\Omega)} \leq C|\boldsymbol{b}_h|_C := C\Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{b}_h\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{m_0}{h_e} \|[\![\boldsymbol{b}_h]\!]_T\|_{0,e}^2\Big)^{1/2}, \tag{1.8}$$

Note here that without any regularity assumption on the domain and with Navier-type boundary conditions both for the velocity and magnetic fields, the solutions $\boldsymbol{u}$ and $\boldsymbol{b}$ of the system (2.1) are both in $\boldsymbol{H}^{1/2}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Therefore, the previous argument cannot be applied to show the continuity of the form C_h. Additional regularity assumptions on the exact solution are therefore necessary. In this work, we show a new discrete inequality analogous to (1.7) on discontinuous spaces provided that the boundary of the domain is sufficiently regular. More precisely, for $1 \leq p \leq 6$, we show that the L^p norm is controlled by the following DG norm :

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^p(\Omega)}^2 \leq C\, \|\boldsymbol{v}_h\|_{v,h}^2 := C\sum_{T\in\mathcal{T}_h} \Big(\|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 + \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2\Big) + \sum_{e\in\mathcal{F}_h} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2\,,$$

This inequality is natural because the DG norm $\|\cdot\|_{v,h}$ is the discrete equivalent of the $\boldsymbol{H}_T^1(\Omega)$ norm. The proof techniques were inspired by the work of Girault and Rivière [19, Lemma 6.2] for the proof of (1.7). However, the proof is not trivial and relies on a regularity result for the second-order elliptic operator $-\Delta = \mathbf{curl}(\mathbf{curl}) - \mathbf{grad}(\operatorname{div})$ with Navier-type boundary conditions in a bounded domain Ω of $\mathbb{R}^3$ and of class $C^{2,1}$.

This new discrete inequality allows, on the one hand, the application of an L^4-L^4-L^2 argument to show the continuity of the form C_h and the form O_h (see (2.5) for the definition of O_h and (1.6) for the definition of C_h) on discrete spaces. On the other hand, it allows us to prove the existence of a solution to the discrete problem using the Brouwer's fixed-point theorem.
The rest of the paper is structured as follows. In Section 2, a discontinuous Galerkin formulation based

on the classical interior penalty (IP) symmetric method of [19] is presented. The main difference is the choice of the stabilization term. Instead of the L^2 norm of the jump of the velocity across elements, we consider stabilization terms based on the normal and tangential jumps of the approximate solutions. The solvability and stability analysis of the discrete problem are established. One key ingredient is the derivation of a new discrete L^p inequality on discontinuous spaces. Convergence results are derived in Section 3 and show that the proposed DG finite element method leads to optimal error bounds for velocity and magnetic field in energy norms.

2 Discontinuous Galerkin finite element approximation

In this section, we will propose a discontinuous Galerkin method for the numerical discretization of the following incompressible MHD model problem : given an externel force $\boldsymbol{f}$ and a source term $\boldsymbol{g}$, we seek a velocity field $\boldsymbol{u}$, a magnetic field $\boldsymbol{b}$ and a scalar pressure p such that

$$-\nu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \nabla p - \kappa(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} = \boldsymbol{f} \qquad \text{in}\,\Omega, \tag{2.1a}$$

$$\kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{g} \qquad \text{in}\,\Omega, \tag{2.1b}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \qquad \text{in}\,\Omega, \tag{2.1c}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on}\,\Gamma, \tag{2.1d}$$

$$\boldsymbol{b}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on}\,\Gamma, \tag{2.1e}$$

where Ω is a bounded simply connected domain in $\mathbb{R}^3$ with a connected boundary Γ. In the model, ν, μ, and κ are respectively the fluid viscosity, magnetic permeability, and coupling term. We refer to [14,17] for further discussion of typical values for these parameters. In addition, we assume $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ satisfying the compatibility condition (1.4).

We will derive solvability and stability results only for the discretized problem. The counterpart for the continuous problem was provided in [31, Theorem 2.3.1]. We note that the following analysis to be developed can be easily extended to the case of nonhomogeneous boundary conditions.

We begin by introducing some notations. We denote by $\mathcal{T}_h$ a regular triangulation of the domain Ω into tetrahedra. The index h is indicative of the mesh size h which is defined as $h = \max\limits_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of T. The family is supposed to be regular in Ciarlet's sense [9], i.e. there exists $\varsigma > 0$ independent of h such that the ratio

$$\frac{h_T}{\rho_T} \leq \varsigma, \quad \forall T \in \mathcal{T}_h, \tag{2.2}$$

where ρ_T is the diameter of the inscribed circle in T. We shall use the assumption (2.2) throughout this work. Let us denote by $\mathcal{F}_h^I$ the set of internal faces and by $\mathcal{F}_h^\Gamma$ the set of external faces on Γ. We set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^\Gamma$. We denote by h_e the diameter of each face e. Let T^+ and T^- be two adjacent elements of $\mathcal{T}_h$ and let $\boldsymbol{n}^+$ (respectively $\boldsymbol{n}^-$) be the outward unit normal vector on ∂T^+ (respectively ∂T^-). For a vector field $\boldsymbol{u}$, we denote by $\boldsymbol{u}^\pm$ the trace of $\boldsymbol{u}$ from the interior of $T^\pm$. We define jumps

$$[\![\boldsymbol{v}]\!]_T := \boldsymbol{n}^+ \times \boldsymbol{v}^+ + \boldsymbol{n}^- \times \boldsymbol{v}^-, \qquad [\![\boldsymbol{v}]\!]_N := \boldsymbol{v}^+\cdot\boldsymbol{n}^+ + \boldsymbol{v}^-\cdot\boldsymbol{n}^-, \quad [\![q]\!] := q^+\boldsymbol{n}^+ + q^-\boldsymbol{n}^-, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}^+ - \boldsymbol{v}^-,$$

and averages

$$\{\!\{\boldsymbol{v}\}\!\} := \frac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-), \qquad \{\!\{q\}\!\} := \frac{1}{2}(q^+ + q^-),$$

and adopt the convention that for boundary faces $e \in \mathcal{F}_h^\Gamma$, we set $[\![\boldsymbol{v}]\!]_T = \boldsymbol{v}\times\boldsymbol{n}$, $[\![\boldsymbol{v}]\!]_N = \boldsymbol{v}\cdot\boldsymbol{n}$, $[\![q]\!] = q\boldsymbol{n}$, $\{\!\{\boldsymbol{v}\}\!\} = \boldsymbol{v}$ and $\{\!\{q\}\!\} = q$.

$\mathcal{P}_k$ denotes the space of polynomials of total degree at most k on T with $k = 1, 2$ or 3. The corresponding vector-valued function space is denoted by $\boldsymbol{\mathcal{P}}_k$.

For $1 \le p \le \infty$, we denote by $L^p(\Omega)$, the set of all functions u defined on Ω such that

$$\int_\Omega |u(x)|^p \,\mathrm{d}x < \infty,$$

equiped with the norm

$$\|u(x)\|_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p \,\mathrm{d}x \Big)^{1/p}.$$

We denote by $W^{m,p}(\Omega)$ the Banach Sobolev space of order m defined by :

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \,:\, D^\alpha u \in L^p(\Omega),\, |\alpha| \le m\},$$

where m is a non-negative integer and D^α denoting a distributional derivative of order $|\alpha|$. The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \Big(\sum_{|\alpha| \le m} \|D^\alpha u\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

We denote by $H^m(\Omega)$, the hilbertian Sobolev space of order $m \ge 0$ instead of $W^{2,m}(\Omega)$.

Throughout this paper, the boldface symbols denote vector-valued quantities. We will use C to denote a generic positive constant independent of h.

2.1 Numerical scheme

We introduce the following finite element spaces which respectively approximate $\boldsymbol{u}$, $\boldsymbol{b}$ and p :

$$\begin{aligned}
\boldsymbol{X}_h &:= \left\{ \boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\, \boldsymbol{v}_{h|T} \in \boldsymbol{\mathcal{P}}_k(T)\,,\, \forall T \in \mathcal{T}_h \right\} \\
\boldsymbol{C}_h &:= \left\{ \boldsymbol{c}_h \in \boldsymbol{L}^2(\Omega);\, \boldsymbol{c}_{h|T} \in \boldsymbol{\mathcal{P}}_k(T)\,,\, \forall T \in \mathcal{T}_h \right\} \\
Q_h &:= \Big\{ q_h \in L^2_0(\Omega);\, q_{h|T} \in \mathcal{P}_{k-1}(T)\,,\, \forall T \in \mathcal{T}_h \Big\}
\end{aligned}$$

We denote by $\boldsymbol{W}_h$ the product space $\boldsymbol{X}_h \times \boldsymbol{C}_h$. The norm $\|\cdot\|_{\boldsymbol{L}^2(\mathcal{T}_h)}$ is defined by

$$\|\cdot\|_{\boldsymbol{L}^2(\mathcal{T}_h)} = \sum_{T \in \mathcal{T}_h} \|\cdot\|_{0,T}\,, \quad \text{for any } T \in \mathcal{T}_h,$$

with $\|\cdot\|_{0,T} = \|\cdot\|_{\boldsymbol{L}^2(T)}$. Similarly, we use the notation $\|\cdot\|_{0,e} = \|\cdot\|_{\boldsymbol{L}^2(e)}$ for any $e \in \mathcal{F}_h$.

The mixed DG scheme reads : Find $((\boldsymbol{u}_h, \boldsymbol{b}_h), p_h) \in \boldsymbol{W}_h \times Q_h$ such that

$$\mathcal{A}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{v}_h, \boldsymbol{c}_h)) + \mathcal{B}_h((\boldsymbol{v}_h, \boldsymbol{c}_h), p_h) = \mathcal{L}_h((\boldsymbol{v}_h, \boldsymbol{c}_h)), \tag{2.3a}$$

$$\mathcal{B}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), q_h) = 0, \tag{2.3b}$$

for all $(\boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$, where the discrete forms $\mathcal{A}_h$ and $\mathcal{B}_h$ are defined by

$$\begin{aligned}
\mathcal{A}_h((\boldsymbol{w}_h, \boldsymbol{d}_h), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{v}_h, \boldsymbol{c}_h)) &= A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + M_h(\boldsymbol{b}_h, \boldsymbol{c}_h) + O_h(\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad + C_h(\boldsymbol{d}_h, \boldsymbol{v}_h, \boldsymbol{b}_h) - C_h(\boldsymbol{d}_h, \boldsymbol{u}_h, \boldsymbol{c}_h), \qquad (2.4a)\\
\mathcal{B}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), q_h) &= B_h(\boldsymbol{u}_h, q_h), \qquad (2.4b)
\end{aligned}$$

for $(\boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{W}_h$, $(\boldsymbol{w}_h, \boldsymbol{d}_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$ all with

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \quad := \quad \nu \sum_{T \in \mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\, \boldsymbol{v}_h \, d\boldsymbol{x} + \nu \sum_{T \in \mathcal{T}_h} \int_T (\mathrm{div}\, \boldsymbol{u}_h)(\mathrm{div}\, \boldsymbol{v}_h)\, d\boldsymbol{x}$$

$$
\begin{aligned}
&- \nu \sum_{e\in\mathcal{F}_h^I} \Big(\int_e \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{v}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds \Big) \\
&- \nu \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{\mathrm{div}\, \boldsymbol{u}_h\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds + \int_e \{\!\{\mathrm{div}\, \boldsymbol{v}_h\}\!\} [\![\boldsymbol{u}_h]\!]_N \, ds \Big) \\
&+ \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_T \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_N [\![\boldsymbol{v}_h]\!]_N \, ds
\end{aligned}
$$

where $\sigma_1, \sigma_2 > 0$ are stabilization parameters that will be chosen large enough to ensure the coercivity of the bilinear form A_h (see Lemma 2.2 below). For the curl-curl term in (2.1a), we apply an augmentation technique where we replace the curl-curl operator by the vector Laplacian (c.f. [20, 30]) : $\mathbf{curl}(\mathbf{curl}) - \mathbf{grad}(\mathrm{div}) = -\Delta$. The two last terms in the definition of A_h involving the tangential and normal jumps of the discrete vector fields across the edges are necessary to ensure the coercivity of the bilinear form A_h (see Lemma 2.2 below). We define the convective term with :

$$
O_h(\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w}_h \times \boldsymbol{u}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x}, \tag{2.5}
$$

for any $\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{X}_h$. We also define the coupling form C_h as :

$$
C_h(\boldsymbol{d}_h, \boldsymbol{v}_h, \boldsymbol{b}_h) := \kappa \sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{v}_h \times \boldsymbol{d}_h) \cdot \mathbf{curl}\, \boldsymbol{b}_h \, d\boldsymbol{x} - \kappa \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{v}_h \times \boldsymbol{d}_h\}\!\} \cdot [\![\boldsymbol{b}_h]\!]_T \, ds, \tag{2.6}
$$

for any $\boldsymbol{v}_h \in \boldsymbol{X}_h$ and $\boldsymbol{d}_h, \boldsymbol{b}_h \in \boldsymbol{C}_h$. The divergence constraint on the velocity is represented by B_h :

$$
B_h(\boldsymbol{v}_h, p_h) := - \sum_{T\in\mathcal{T}_h} \int_T (\mathrm{div}\, \boldsymbol{v}_h) p_h \, d\boldsymbol{x} + \sum_{e\in\mathcal{F}_h} \int_e \{\!\{p_h\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds. \tag{2.7}
$$

The form M_h is defined by :

$$
\begin{aligned}
M_h(\boldsymbol{b}_h, \boldsymbol{c}_h) \;:=\; & \kappa\mu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{b}_h \cdot \mathbf{curl}\, \boldsymbol{c}_h \, d\boldsymbol{x} + \kappa\mu \sum_{T\in\mathcal{T}_h} \int_T (\mathrm{div}\, \boldsymbol{b}_h)(\mathrm{div}\, \boldsymbol{c}_h) \, d\boldsymbol{x} \\
& - \kappa\mu \sum_{e\in\mathcal{F}_h^I} \Big(\int_e \{\!\{\mathbf{curl}\, \boldsymbol{b}_h\}\!\} \cdot [\![\boldsymbol{c}_h]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{c}_h\}\!\} \cdot [\![\boldsymbol{b}_h]\!]_T \, ds \Big) \\
& - \kappa\mu \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{\mathrm{div}\, \boldsymbol{b}_h\}\!\} [\![\boldsymbol{c}_h]\!]_N \, ds + \{\!\{\mathrm{div}\, \boldsymbol{c}_h\}\!\} [\![\boldsymbol{b}_h]\!]_N \, ds \Big) \\
& + \sum_{e\in\mathcal{F}_h^I} \frac{\kappa\mu m_1}{h_e} \int_e [\![\boldsymbol{b}_h]\!]_T \cdot [\![\boldsymbol{c}_h]\!]_T \, ds + \sum_{e\in\mathcal{F}_h} \frac{\kappa\mu m_2}{h_e} \int_e [\![\boldsymbol{b}_h]\!]_N [\![\boldsymbol{c}_h]\!]_N \, ds
\end{aligned}
$$

with $m_1, m_2 > 0$ are stabilization parameters that will be chosen large enough to ensure the coercivity of the bilinear form M_h (see Lemma 2.2 below). Analogous to A_h, we have applied an augmentation technique for the curl-curl term in the equation (2.1b). Finally, the form $\mathcal{L}_h$ is defined by

$$
\mathcal{L}_h(\boldsymbol{v}_h, \boldsymbol{c}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, d\boldsymbol{x} + \int_\Omega \boldsymbol{g} \cdot \boldsymbol{c}_h \, d\boldsymbol{x}.
$$

Remark 2.1 *Another discontinuous Galerkin formulation is possible to solve the MHD problem (2.1). Indeed, the curl-curl operator can be discretized using a standard interior penalty approach (see [22, 25, 32]) and then the form M_h is defined as*

$$
M_h(\boldsymbol{b}_h, \boldsymbol{c}_h) \;:=\; \kappa\mu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{b}_h \cdot \mathbf{curl}\, \boldsymbol{c}_h \, d\boldsymbol{x} - \kappa\mu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{b}_h\}\!\} \cdot [\![\boldsymbol{c}_h]\!]_T \, ds
$$

$$- \quad \kappa\mu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{c}_h\}\!\} \cdot [\![\boldsymbol{b}_h]\!]_T \, ds + \sum_{e\in\mathcal{F}_h^I} \frac{\kappa\mu m_0}{h_e} \int_e [\![\boldsymbol{b}_h]\!]_T \cdot [\![\boldsymbol{c}_h]\!]_T \, ds \tag{2.8}$$

with $m_0 > 0$ a stabilization parameter sufficiently large. The divergence-free constraint for the magnetic field is imposed by introducing a Lagrange multiplier r_h. This approach is motivated by the lack of the regularity of $\boldsymbol{b}$ when Ω is not smooth. As mentionned above, if Ω is non-convex polyhedra, the magnetic field $\boldsymbol{b}$ belongs to $\boldsymbol{H}^{1/2}(\Omega)$ only. Due to our assumption on Ω, the magnetic field as well as the velocity field have the regularity $\boldsymbol{H}^1(\Omega)$.

To discuss the stability properties of the above forms, we need to introduce the following semi-norms

$$\|\boldsymbol{u}_h\|_{v,h}^2 := |\boldsymbol{u}_h|_{1,h}^2 + \sum_{e\in\mathcal{F}_h} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 , \qquad \forall \boldsymbol{u}_h \in \boldsymbol{X}_h.$$

$$\|\boldsymbol{b}_h\|_{m,h}^2 := |\boldsymbol{b}_h|_{1,h}^2 + \sum_{e\in\mathcal{F}_h} \frac{m_1}{h_e} \|[\![\boldsymbol{b}_h]\!]_N\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{m_2}{h_e} \|[\![\boldsymbol{b}_h]\!]_T\|_{0,e}^2 , \qquad \forall \boldsymbol{b}_h \in \boldsymbol{C}_h.$$

where

$$|\cdot|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\mathrm{div} \cdot\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\mathbf{curl} \cdot\|_{0,T}^2 ,$$

and

$$\|q_h\|_{Q_h}^2 := \|q_h\|_{L^2(\Omega)}^2 , \qquad \forall q_h \in Q_h.$$

We note that the semi-norm $\|\cdot\|_{v,h}$ (respectively $\|\cdot\|_{m,h}$) actually defines a norm on $\boldsymbol{X}_h$ (respectively on $\boldsymbol{C}_h$). (See [28, Remark 3.3]

Next, we recall the following discrete Poincaré-Friedrichs inequality for discontinuous finite element spaces (see [42, Lemma 3.1]) :

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C \|\boldsymbol{v}_h\|_{v,h} , \tag{2.9}$$

where the constant $C > 0$ is independent of h. Since $\boldsymbol{X}_h = \boldsymbol{C}_h$, this inequality is still valid for any $\boldsymbol{b}_h \in \boldsymbol{C}_h$ and when the norm $\|\cdot\|_{v,h}$ is replaced by $\|\cdot\|_{m,h}$.

2.2 A new L^p discrete Sobolev's inequality on discountinuous spaces

In this subsection, we first establish a regularity result for the Laplace's equation with Navier-type boundary condition. This result plays a central role in the proof of a new discrete Sobolev embedding allowing one to establish the well-posedness and the convergence of the DG scheme (2.3). In the following result, we demonstrate that it is indeed possible to derive a regularity result similar to those in [4, 5] for the Stokes problem when the divergence constraint is not imposed. This result is presented here in a non Hilbertian setting which is more general than that needed to analyze our model. However, since the result below is of independent interest to analyze other nonlinear problems, we choose to give it in $\boldsymbol{L}^p$ spaces with $p \geq 6/5$.

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^3$ be an open bounded simply-connected set of class $\mathcal{C}^{2,1}$. Let us suppose that $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ with $p \geq 6/5$. Then, the following Laplace equation :*

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{g} & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma \end{cases} \tag{2.10}$$

has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ which also satisfies :

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leqslant C \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} . \tag{2.11}$$

Proof. We first consider the case $p = 2$ and prove the existence and uniqueness of solutions in $\boldsymbol{H}^1(\Omega)$. The problem (2.10) is equivalent to the following variational formulation : Find $\boldsymbol{u} \in \boldsymbol{H}^1_T(\Omega)$ such that :

$$\forall\, \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega), \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega (\mathrm{div} \boldsymbol{u})\, (\mathrm{div} \boldsymbol{v})\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{g} \cdot \boldsymbol{v}\, d\boldsymbol{x}, \tag{2.12}$$

where $\boldsymbol{H}^1_T(\Omega)$ is the space of functions in $\boldsymbol{H}^1(\Omega)$ with zero normal trace defined in (1.5). Let us introduce the bilinear continuous form : $a(\cdot,\cdot) : \boldsymbol{H}^1_T(\Omega) \times \boldsymbol{H}^1_T(\Omega) \to \mathbb{R}$ defined as follows :

$$a(\boldsymbol{u},\, \boldsymbol{v}) = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega (\mathrm{div} \boldsymbol{u})\, (\mathrm{div} \boldsymbol{v})\, d\boldsymbol{x}.$$

Using the following Poincaré inequality [18, Theorem 3.9],

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C_P\big(\, \|\mathbf{curl}\ \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)}\, \big), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega), \tag{2.13}$$

the form $a(\cdot,\cdot)$ is coercive on $\boldsymbol{H}^1_T(\Omega)$. Since the right hand-side defines a linear continuous form on $\boldsymbol{H}^1_T(\Omega)$, we deduce by the Lax-Milgram's Lemma that problem (2.12) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1_T(\Omega)$ satisfying the estimate :

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\, \boldsymbol{g}\, \|_{\boldsymbol{L}^2(\Omega)}. \tag{2.14}$$

Moreover, we set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. Then $\boldsymbol{z}$ satisfies the problem

$$\begin{cases} -\Delta\, \boldsymbol{z} = \mathbf{curl}\, \boldsymbol{g} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{2.15}$$

Since The right hand side in (2.15) is a curl of function in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g}$ satisfies the compatibility condition $\mathrm{div}(\mathbf{curl}\ \boldsymbol{g}) = 0$ in Ω, it follows from [31, Lemma 1.3.4](see also [5, Theorem 5.7]) that $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega)$ and satisfies the estimate

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = \|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}. \tag{2.16}$$

As a consequence, $\nabla\, \mathrm{div}\, \boldsymbol{u} = \Delta \boldsymbol{u} + \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ and then $\mathrm{div}\, \boldsymbol{u}$ belongs to $H^1(\Omega)$. Applying [5, Corollary 3.5] leads to deduce that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$. Using (2.16), we obtain

$$\|\mathrm{div}\ \boldsymbol{u}\|_{H^1(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}. \tag{2.17}$$

Finally, estimate (2.11) for $p = 2$ follows from (2.16) and (2.17). For the result in the non Hilbert case, we consider two cases.

First case : $p > 2$. We know that problem (2.10) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Then $\boldsymbol{u}$ belongs to $\boldsymbol{L}^q(\Omega)$ for all $1 \leq q \leq \infty$. Moreover, since the right hand side in (2.15) is a curl of function in $\boldsymbol{L}^p(\Omega)$, it follows again from [31, Lemma 1.3.4] (see also [5, Theorem 5.7]) that problem (2.15) has a unique solution $\boldsymbol{z} = \mathbf{curl}\ \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Similarly to the case $p = 2$, this implies that $\nabla\, \mathrm{div}\, \boldsymbol{u}$ belongs to $\boldsymbol{L}^p(\Omega)$ and by [3, Proposition 2.10], we deduce that $\mathrm{div}\, \boldsymbol{u} \in W^{1,p}(\Omega)$. In summary, we have

$$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega), \quad \mathrm{div}\, \boldsymbol{u} \in L^p(\Omega), \quad \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma. \tag{2.18}$$

Applying again [5, Corollary 3.5], we deduce that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (2.11).

Second case : $6/5 \leq p < 2$. Observe that $\boldsymbol{g}$ is at least $\boldsymbol{L}^{6/5}(\Omega)$. Since $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, the right-hand side in (2.12) still defines a linear continuous form on $\boldsymbol{H}^1(\Omega)$. By the Lax-Milgram lemma, the problem (2.10) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisfying the estimate :

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\, \boldsymbol{g}\, \|_{\boldsymbol{L}^{6/5}(\Omega)}.$$

Next, we use the same argument as in the first step in order to prove that $\mathbf{curl}\, \boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and then $\nabla\, \mathrm{div}\, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Now, since $\boldsymbol{u} \in \boldsymbol{L}^6(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$, the solution $\boldsymbol{u}$ satisfies (2.18). Applying again [5, Corollary 3.5], we deduce that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (2.11). $\square$

To prove that the discrete problem is well posed, we shall require a discrete $\boldsymbol{L}^p$ estimate for functions in $\boldsymbol{X}_h$ with $p \leq 6$, in terms of the norm $\|\cdot\|_{v,h}$. The inequality (2.20) given in the next Lemma is the equivalent of the following $\boldsymbol{L}^r$ Sobolev's inequality with $r \in [2, \infty)$ proven in [19, Lemma 6.2] in two dimensions : for any $\boldsymbol{v}_h \in \boldsymbol{X}_h$,

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^r(\Omega)} \leq C(r)|\!|\!|\boldsymbol{v}_h|\!|\!|_h := C(r)\Big(\sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h\|^2_{0,T} + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{h_e} \|[\![\boldsymbol{v}_h]\!]\|^2_{0,e} \Big)^{1/2}, \tag{2.19}$$

where $\gamma > 0$ is a stabilization parameter.
Sobolev's inequality (2.20) will be needed in the proof of Theorem 2.6 below. More specifically, it is required to bound the forms O_h in terms of the norm $\|\cdot\|_{v,h}$ when deriving the continuity of the operator $\mathcal{F}_h$. This inequality will also be required to handle the coupling form C_h. The proof is similar to (2.19) but not elementary because it heavily relies on the new regularity theorem stated in Proposition 2.1. So, for the rest of this paper, we shall make an additional smoothness assumptions on Ω. Indeed, we suppose that the domain Ω has a boundary of class $\mathcal{C}^{2,1}$.

Lemma 2.1 *For each real number $p \in (1, 6]$, there exists a constant $C > 0$ independent of h such that :*

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^p(\Omega)} \leqslant C \|\boldsymbol{v}_h\|_{v,h} \tag{2.20}$$

Proof. Following [19], we introduce the lifting $\boldsymbol{v}(h)$ of $\boldsymbol{v}_h \in \boldsymbol{X}_h$ where $\boldsymbol{v}(h) \in \boldsymbol{H}^1_T(\Omega)$ is the unique solution of

$$\int_\Omega \mathbf{curl}\, \boldsymbol{v}(h) \cdot \mathbf{curl}\, \boldsymbol{w}\, d\boldsymbol{x} + \int_\Omega \operatorname{div} \boldsymbol{v}(h) \operatorname{div} \boldsymbol{w}\, d\boldsymbol{x} = \sum_{T\in\mathcal{T}_h} \Big(\int_T \mathbf{curl}\, \boldsymbol{v}_h \cdot \mathbf{curl}\, \boldsymbol{w}\, d\boldsymbol{x} + \int_T (\operatorname{div} \boldsymbol{v}_h)(\operatorname{div} \boldsymbol{w})\, d\boldsymbol{x} \Big),$$

for any $\boldsymbol{w} \in \boldsymbol{H}^1_T(\Omega)$. Moreover, we have the following estimate

$$\|\operatorname{div} \boldsymbol{v}(h)\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}(h)\|_{\boldsymbol{L}^2(\Omega)} \leq \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|^2_{0,T} + \sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|^2_{0,T} \Big)^{1/2} \leq \|\boldsymbol{v}_h\|_{v,h}\, .$$

Since $\boldsymbol{v}(h)$ belongs to $\boldsymbol{H}^1_T(\Omega)$, we have from (2.13) for any $1 \leq p \leq 6$

$$\|\boldsymbol{v}(h)\|_{\boldsymbol{L}^p(\Omega)} \leq C \|\boldsymbol{v}_h\|_{v,h}\, . \tag{2.21}$$

So, it suffices to prove that (2.20) holds for $\boldsymbol{v}_h - \boldsymbol{v}(h)$ and then to use triangle inequality. By duality, we have

$$\|\boldsymbol{v}_h - \boldsymbol{v}(h)\|_{\boldsymbol{L}^p(\Omega)} = \sup_{\boldsymbol{g}\in\boldsymbol{L}^{p'}(\Omega)} \frac{\displaystyle\int_\Omega (\boldsymbol{v}_h - \boldsymbol{v}(h)) \cdot \boldsymbol{g}\, d\boldsymbol{x}}{\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)}},$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$. Since $p \leqslant 6$, then $p' \geqslant \frac{6}{5}$ and $\boldsymbol{g}$ always belongs to $\boldsymbol{L}^{6/5}(\Omega)$. Thanks to Proposition 2.1, the following problem

$$\begin{cases} -\Delta \boldsymbol{\phi} = \boldsymbol{g} & \text{in } \Omega \\ \mathbf{curl}\, \boldsymbol{\phi} \times \boldsymbol{n} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\phi} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \end{cases}$$

has a unique solution $\boldsymbol{\phi} \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega)$ satisfying the estimate

$$\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{2,\frac{6}{5}}(\Omega)} \leqslant C \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)} \tag{2.22}$$

Then, since $[\![\boldsymbol{v}(h)]\!]_N = 0$ for all $e \in \mathcal{F}_h$, $[\![\boldsymbol{v}(h)]\!]_T = \mathbf{0}$ for all $e \in \mathcal{F}^I_h$ and $(\mathbf{curl}\, \boldsymbol{\phi}) \times \boldsymbol{n} = \mathbf{0}$ on Γ, the regularity of $\boldsymbol{\phi}$ and (2.21) imply that

$$\int_\Omega (\boldsymbol{v}_h - \boldsymbol{v}(h)) \cdot \boldsymbol{g}\, d\boldsymbol{x} \quad = \quad -\int_\Omega \Delta \boldsymbol{\phi} \cdot (\boldsymbol{v} - \boldsymbol{v}(h))\, d\boldsymbol{x}$$

$$= \quad -\sum_{e\in\mathcal{F}_h}\int_e \operatorname{div}\boldsymbol{\phi}\,[\![\boldsymbol{v}_h]\!]_N\,ds + \sum_{e\in\mathcal{F}_h^I}\int_e \mathbf{curl}\,\boldsymbol{\phi}\cdot[\![\boldsymbol{v}_h]\!]_T\,ds. \tag{2.23}$$

We need to bound the right-hand side of (2.23). We give the proof for the first term involving the divergence oprator and the normal jumps. A similar bound for the second term, involving the rotational operator and the tangential jumps, can be obtained exactly with the same arguments and obvious modifications. Since $\operatorname{div}\boldsymbol{\phi}$ belongs to $\boldsymbol{W}^{1,6/5}(\Omega)$, the trace of $\operatorname{div}\boldsymbol{\phi}$ on each face e belongs to $\boldsymbol{W}^{1/6,6/5}(e)\hookrightarrow\boldsymbol{L}^{4/3}(e)$. Then, we have

$$\Big|\int_e \operatorname{div}\boldsymbol{\phi}\,[\![\boldsymbol{v}_h]\!]_N\,ds\Big| \leqslant \|\operatorname{div}\boldsymbol{\phi}\|_{\boldsymbol{L}^{4/3}(e)}\,\|[\![\boldsymbol{v}_h]\!]_N\|_{L^4(e)}\,. \tag{2.24}$$

Passing to the reference element $\hat{T}$ with the face $\hat{e}$, we have

$$\|\operatorname{div}\boldsymbol{\phi}\|_{\boldsymbol{L}^{4/3}(e)} \le C|e|^{\frac{3}{4}}\left\|\widehat{\operatorname{div}\boldsymbol{\phi}}\right\|_{L^{4/3}(\hat{e})}.$$

Next, using the following trace inequality :

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(e)} \le c h_T^{-\frac{1}{p}+d(\frac{1}{p}-\frac{1}{q})}\,\|\boldsymbol{v}\|_{\boldsymbol{L}^q(T)}\,, \tag{2.25}$$

with $p=\frac{4}{3}$, $q=2$ and $d=3$ together with the embedding $W^{1,6/5}(\hat{T})\hookrightarrow L^2(\hat{T})$, we obtain

$$\|\operatorname{div}\boldsymbol{\phi}\|_{L^{4/3}(e)} \le C|e|^{\frac{3}{4}}\left\|\widehat{\operatorname{div}\boldsymbol{\phi}}\right\|_{L^2(\hat{T})} \le C|e|^{\frac{3}{4}}\left\|\widehat{\operatorname{div}\boldsymbol{\phi}}\right\|_{W^{1,6/5}(\hat{T})}.$$

Applying the inequality

$$\|\widehat{v}\|_{W^{1,p}(\hat{T})} \le C\frac{h_T}{|T|^{\frac{1}{p}}}\,\|v\|_{W^{1,p}(T)}\,,$$

we deduce that

$$\|\operatorname{div}\boldsymbol{\phi}\|_{L^{4/3}(e)} \le C|e|^{\frac{3}{4}}\frac{h_T}{|T|^{\frac{5}{6}}}\,\|\operatorname{div}\boldsymbol{\phi}\|_{W^{1,6/5}(T)}\,.$$

Moreover, by virtue of assumption (2.2), we have for any $T\in\mathcal{T}_h$

$$\frac{|e|}{\sqrt{T}} \le C\frac{h_T}{\rho_T} \le C\varsigma,$$

which gives

$$\|\operatorname{div}\boldsymbol{\phi}\|_{L^{4/3}(e)} \le C\,\|\operatorname{div}\boldsymbol{\phi}\|_{W^{1,6/5}(T)}\,. \tag{2.26}$$

On the other hand, we have

$$\|[\![\boldsymbol{v}_h]\!]_N\|_{L^4(e)} \le C|e|^{1/4}\,\|[\![\widehat{\boldsymbol{v}_h}]\!]_N\|_{L^2(\hat{e})} \le C|e|^{-1/2}\,\|[\![\boldsymbol{v}_h]\!]_N\|_{L^2(e)}\,, \tag{2.27}$$

where a local equivalence of norms is used. Combining (2.26) and (2.27) in (2.24), we obtain

$$\Big|\int_e \operatorname{div}\boldsymbol{\phi}\,[\![\boldsymbol{v}_h]\!]_N\,ds\Big| \leqslant C\,\|\operatorname{div}\boldsymbol{\phi}\|_{W^{1,6/5}(T)}\,|e|^{-1/2}\,\|[\![\boldsymbol{v}_h]\!]_N\|_{L^2(e)}\,. \tag{2.28}$$

Finally, summing over $e\in\mathcal{F}_h$ and using estimate (2.22), we obtain

$$\sum_{e\in\mathcal{F}_h}\int_e \operatorname{div}\boldsymbol{\phi}\,[\![\boldsymbol{v}_h]\!]_N\,ds \leqslant C\,\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)}\Big(\sum_{e\in\mathcal{F}_h}\frac{\sigma_1}{|e|}\,\|[\![\boldsymbol{v}_h]\!]_N\|^2_{L^2(e)}\Big)^{1/2}. \tag{2.29}$$

A very much similar proof gives

$$\sum_{e\in\mathcal{F}_h^I}\int_e \mathbf{curl}\,\boldsymbol{\phi}\cdot[\![\boldsymbol{v}_h]\!]_T\,ds \leqslant C\,\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)}\Big(\sum_{e\in\mathcal{F}_h^I}\frac{\sigma_2}{|e|}\,\|[\![\boldsymbol{v}_h]\!]_T\|^2_{L^2(e)}\Big)^{1/2}. \tag{2.30}$$

Collecting the above estimates (2.29)-(2.30) into (2.23), we obtain

$$\int_\Omega(\boldsymbol{v}_h-\boldsymbol{v}(h))\cdot\boldsymbol{g}\,d\boldsymbol{x}\quad\leqslant C\,\|\boldsymbol{g}\|_{L^{p'}(\Omega)}\,\|\boldsymbol{v}_h\|_{v,h}$$

which achieves the proof of the estimate (2.20). □

Remark 2.2 *We recall the following Sobolev embedding $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^p(\Omega)$ for all real numbers $1\le p\le 6$. So, we have from (2.13)*

$$\forall\boldsymbol{v}\in\boldsymbol{H}^1_T(\Omega),\quad \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\le C\big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}+\|\mathrm{div}\boldsymbol{v}\|_{L^2(\Omega)}\big). \tag{2.31}$$

Observe that the discrete Sobolev embedding (2.20) is the counterpart of that valid at the continuous level (2.31). Indeed, if $\boldsymbol{v}\in\boldsymbol{H}^1_T(\Omega)$, we have $[\![\boldsymbol{v}(h)]\!]_N=0$ for all $e\in\mathcal{F}_h$ and $[\![\boldsymbol{v}(h)]\!]_T=\mathbf{0}$ for all $e\in\mathcal{F}_h^I$ and then, we have (2.31).

2.3 Continuity and Ellipticity properties

In this section, we state and discuss the main properties of the forms involved in our discrete scheme (2.3). We have a first lemma which ensures the coercivity for the forms A_h and M_h. Even though the proof is the same of Lemma 10.2.1 of [36] for the case of the two-dimensional Stokes problem, we would rather present full details for the sake of completeness

Lemma 2.2 *For σ_1 and σ_2 large enough, there exists a positive constant C independent of h such that :*

$$A_h(\boldsymbol{u}_h,\boldsymbol{u}_h)\geqslant C\nu\,\|\boldsymbol{u}_h\|^2_{v,h}\,,\qquad\forall\boldsymbol{u}_h\in\boldsymbol{X}_h. \tag{2.32}$$

For m_1 and m_2 large enough, there exists a positive constant C independent of h such that :

$$M_h(\boldsymbol{b}_h,\boldsymbol{b}_h)\geqslant C\kappa\mu\,\|\boldsymbol{b}_h\|^2_{m,h}\,,\qquad\forall\boldsymbol{b}_h\in\boldsymbol{C}_h. \tag{2.33}$$

Besides, we have $O_h(\boldsymbol{w}_h,\boldsymbol{u}_h,\boldsymbol{u}_h)=0$ for any $\boldsymbol{w}_h$, $\boldsymbol{u}_h$, $\boldsymbol{v}_h\in\boldsymbol{X}_h$.

Proof. Trivially, since the vector $\mathbf{curl}\,\boldsymbol{w}_h\times\boldsymbol{v}_h$ is orthogonal to $\boldsymbol{v}_h$, its scalar product with $\boldsymbol{v}_h$ is zero. So, it follows from the definition of O_h :

$$O_h(\boldsymbol{w}_h,\boldsymbol{u}_h,\boldsymbol{u}_h)=\sum_{T\in\mathcal{T}_h}\int_T(\mathbf{curl}\,\boldsymbol{w}_h\times\boldsymbol{u}_h)\cdot\boldsymbol{u}_h\,d\boldsymbol{x}=0.$$

Now, we prove the ellipticity property for A_h. Let $\boldsymbol{u}_h\in\boldsymbol{X}_h$, we have :

$$\begin{aligned}A_h(\boldsymbol{u}_h,\boldsymbol{u}_h)=\nu|\boldsymbol{u}_h|^2_{1,h}&-2\nu\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\mathbf{curl}\,\boldsymbol{u}_h\}\!\}\cdot[\![\boldsymbol{u}_h]\!]_T\,ds-2\nu\sum_{e\in\mathcal{F}_h}\int_e\{\!\{\mathrm{div}\,\boldsymbol{u}_h\}\!\}[\![\boldsymbol{u}_h]\!]_N\,ds\\&+\nu\sum_{e\in\mathcal{F}_h^I}\frac{\sigma_1}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_T\|^2_{0,e}+\nu\sum_{e\in\mathcal{F}_h}\frac{\sigma_2}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_N\|^2_{0,e}\end{aligned} \tag{2.34}$$

Applying the Cauchy-Schwarz inequality, we have :

$$2\nu\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\mathbf{curl}\,\boldsymbol{u}_h\}\!\}\cdot[\![\boldsymbol{u}_h]\!]_T\,ds\leqslant 2\nu\Big(\sum_{e\in\mathcal{F}_h^I}h_e\,\|\{\!\{\mathbf{curl}\,\boldsymbol{u}_h\}\!\}\|^2_{0,e}\Big)^{\frac12}\Big(\sum_{e\in\mathcal{F}_h^I}h_e^{-1}\,\|[\![\boldsymbol{u}_h]\!]_T\|^2_{0,e}\Big)^{\frac12}$$

Now let $e \in \mathcal{F}_h^I$ such that $e = \partial T_1 \cap \partial T_2$ with $T_1, T_2 \in \mathcal{T}_h$. Obviously,

$$\sqrt{h_e}\| \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\}\|_{0,e} \;\; \le \;\; \frac{\sqrt{h_e}}{2}\sum_{i=1}^{2}\|(\mathbf{curl}\;\boldsymbol{u}_h)_{/T_i}\|_{0,e}.$$

Thanks to the equivalence of norms in finite dimensional spaces and to a classical scaling argument, we obtain

$$\sqrt{h_e}\| \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\}\|_{0,e} \;\; \le \;\; \frac{c_1}{2}\Big(\|\,\mathbf{curl}\,\boldsymbol{u}_h\|_{0,\mathrm{T}_1} + \|\,\mathbf{curl}\,\boldsymbol{u}_h\|_{0,\mathrm{T}_2}\Big),$$

where c_1 is a constant independent of the discretization. A completely similar argument holds on a boundary face $e \in \mathcal{F}_h^\Gamma$. So by summing upon all faces it follows that

$$\begin{aligned} 2\nu\sum_{e\in\mathcal{F}_h^I}\int_e \{\!\{\mathbf{curl}\,\boldsymbol{u}_h\}\!\}\cdot[\![\boldsymbol{u}_h]\!]_T\,ds \;\; &\le \;\; 2c_1\nu\Big(\sum_{T\in\mathcal{T}_h}\|\,\mathbf{curl}\;\boldsymbol{u}_h\|_{0,T}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1}\,\|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2\Big)^{\frac{1}{2}} \\ &\le \;\; 2c_1\nu|\boldsymbol{u}_h|_{1,h}\Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1}\,\|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Similarly, we can check that

$$2\nu\sum_{e\in\mathcal{F}_h}\int_e \{\!\{\mathrm{div}\,\boldsymbol{u}_h\}\!\}[\![\boldsymbol{u}_h]\!]_N\,ds \le 2c_2\nu|\boldsymbol{u}_h|_{1,h}\Big(\sum_{e\in\mathcal{F}_h} h_e^{-1}\,\|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2\Big)^{\frac{1}{2}}.$$

As a consequence, we have

$$\begin{aligned} A_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \;\ge\; & \nu|\boldsymbol{u}_h|_{1,h}^2 + \nu\sum_{e\in\mathcal{F}_h^I}\frac{\sigma_1}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 + \nu\sum_{e\in\mathcal{F}_h}\frac{\sigma_2}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2 \\ & -\frac{2c_1\nu}{\sqrt{\sigma_1}}|\boldsymbol{u}_h|_{1,h}\Big(\sum_{e\in\mathcal{F}_h^I}\frac{\sigma_1}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2\Big)^{\frac{1}{2}} - \frac{2c_2\nu}{\sqrt{\sigma_2}}|\boldsymbol{u}_h|_{1,h}\Big(\sum_{e\in\mathcal{F}_h}\frac{\sigma_2}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned} A_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \;\;\ge\;\; & \nu|\boldsymbol{u}_h|_{1,h}^2 + \nu\sum_{e\in\mathcal{F}_h^I}\frac{\sigma_1}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 + \nu\sum_{e\in\mathcal{F}_h}\frac{\sigma_2}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2 \\ & - \frac{c\,\nu}{\sigma}\Big(|\boldsymbol{u}_h|_{1,h}^2 + \sum_{e\in\mathcal{F}_h^I}\frac{\sigma_1}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h}\frac{\sigma_2}{h_e}\,\|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2\Big), \end{aligned}$$

where $c = c_1 + c_2$ and $\sigma = \min(\sqrt{\sigma_1},\, \sqrt{\sigma_2})$. So $A_h(\cdot,\cdot)$ is coercive for $\sigma > c$ and the coercivity constant C in (2.32) can be obtained from below independently of σ_1 and σ_2, for σ_1 and σ_2 sufficiently large. With the same manner, we prove the ellipticity property of M_h by using the definition of $\|\boldsymbol{b}_h\|_{m,h}$ and by supposing that the parameters m_1 and m_2 are sufficiently large. □

Remark 2.3 *Compared with the approach in [22, 25, 32], a semi-coercivity property for M_h, defined in (2.8), is established with respect to the semi-norm*

$$|\boldsymbol{b}_h|_C^2 := \sum_{T\in\mathcal{T}_h}\|\mathbf{curl}\,\boldsymbol{b}_h\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h^I}\frac{m_0}{h_e}\,\|[\![\boldsymbol{b}_h]\!]_T\|_{0,e}^2\,, \tag{2.35}$$

for a parameter m_0 sufficiently large.

Now, we give the following continuity results with respect of the norms $\|\cdot\|_{v,h}$ and $\|\cdot\|_{m,h}$. The proof is similar to that in [28, Proposition 3.6] for the form A_h and to that in [10, 24] for the form B_h, but we write it here for the reader's convenience.

Lemma 2.3 *Let $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{X}_h$, $\boldsymbol{b}_h, \boldsymbol{c}_h \in \boldsymbol{C}_h$ and $p_h \in Q_h$. Then, we have :*

$$\begin{aligned}
|A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)| &\leqslant C\nu \|\boldsymbol{u}_h\|_{v,h} \|\boldsymbol{v}_h\|_{v,h}, \\
|B_h(\boldsymbol{v}_h, p_h)| &\leqslant C \|\boldsymbol{v}_h\|_{v,h} \|p_h\|_{0,\Omega}, \\
|M_h(\boldsymbol{b}_h, \boldsymbol{c}_h)| &\leqslant C\kappa\mu \|\boldsymbol{b}_h\|_{m,h} \|\boldsymbol{c}_h\|_{m,h},
\end{aligned}$$

with constants $C > 0$ that are independent of h, ν, μ and κ.

Proof. Let us write : $A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := I_1 + I_2 + I_3 + I_4$, with

$$\begin{aligned}
I_1 &:= \nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\ \boldsymbol{v}_h \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div} \boldsymbol{u}_h \operatorname{div} \boldsymbol{v}_h \, d\boldsymbol{x} \\
I_2 &:= -\nu \sum_{e\in\mathcal{F}_h^I} \Big(\int_e \{\!\{\mathbf{curl}\ \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{v}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds \Big) \\
I_3 &:= -\nu \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{\operatorname{div}\ \boldsymbol{u}_h\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds + \int_e \{\!\{\operatorname{div} \boldsymbol{v}_h\}\!\} [\![\boldsymbol{u}_h]\!]_N \, ds \Big) \\
I_4 &:= \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_T \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_N [\![\boldsymbol{v}_h]\!]_N \, ds.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|I_1| &\leqslant \nu \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 \Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac{1}{2}} + \nu \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{u}_h\|_{0,T}^2 \Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac{1}{2}} \\
&\le \nu \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{u}_h\|_{0,T}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 \Big)^{1/2} \\
&\le \nu \|\boldsymbol{u}_h\|_{v,h} \|\boldsymbol{v}_h\|_{v,h}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|I_4| &\leqslant \nu \Big(\sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{v}_h]\!]_T\|^2 \Big)^{\frac{1}{2}} + \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|^2 \Big)^{\frac{1}{2}} \\
&\le \nu \|\boldsymbol{u}_h\|_{v,h} \|\boldsymbol{v}_h\|_{v,h}.
\end{aligned}$$

Now, applying Cauchy-Schwarz inequality and the inverse inequality, we obtain :

$$\begin{aligned}
|I_2| &\leqslant \nu C \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \|[\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}} + \nu C \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}} \\
&\le C\nu \|\boldsymbol{u}_h\|_{v,h} \|\boldsymbol{v}_h\|_{v,h}.
\end{aligned}$$

In a similar way, for T_3, we can verify that $|I_3| \le C\nu \|\boldsymbol{u}_h\|_{v,h} \|\boldsymbol{v}_h\|_{v,h}$. Adding these previous estimates, we deduce the continuity for A_h. To prove the continuity of the form M_h it suffices to use the same techniques as in the proof for A_h. Next, the form B_h is similar to the form used in [10, 24] for the Stokes system and we can prove similarly

$$\begin{aligned}
|B_h(\boldsymbol{v}_h, p_h)| &\leqslant \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|p_h\|_{0,T}^2 \Big)^{\frac{1}{2}} + \Big(\sum_{e\in\mathcal{F}_h} \frac{h_e}{\sigma_2} \|\{\!\{p_h\}\!\}\|_{0,e}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|_{0,e}^2 \Big)^{\frac{1}{2}} \\
&\leqslant \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|p_h\|_{0,T}^2 \Big)^{\frac{1}{2}} + C\sigma_2^{-\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|p_h\|_{0,T}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|_{0,e}^2 \Big)^{\frac{1}{2}} \\
&\leqslant C \|\boldsymbol{v}_h\|_{v,h} \|p_h\|_{0,\Omega}.
\end{aligned}$$

□

Our next goal is to establish the continuity properties for the non linear forms O_h and C_h. In view of (2.20), we have the following result concerning the continuity of the form O_h.

Lemma 2.4 *There exists a constant $C > 0$ independent of h such that for any $\boldsymbol{w}_h$, $\boldsymbol{u}_h$, $\boldsymbol{v}_h \in \boldsymbol{X}_h$,*

$$O_h(\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \leqslant C\,\|\boldsymbol{w}_h\|_{v,h}\,\|\boldsymbol{u}_h\|_{v,h}\,\|\boldsymbol{v}_h\|_{v,h}\,. \tag{2.36}$$

Proof. Using Hölder's inequality, we obtain

$$O_h(\boldsymbol{w}_h, \boldsymbol{u}_h,\, \boldsymbol{v}_h) := \sum_{T\in\mathcal{T}_h}\int_T (\mathbf{curl}\,\boldsymbol{w}_h \times \boldsymbol{u}_h)\cdot \boldsymbol{v}_h\, d\boldsymbol{x} \leq C\Big(\sum_{T\in\mathcal{T}_h}\|\,\mathbf{curl}\,\boldsymbol{w}_h\|^2_{0,T}\Big)^{1/2}\|\boldsymbol{u}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}$$

Applying (2.20) with $p = 4$, the result is then achieved. □

For the continuity of the coupling form C_h, we require the discrete Sobolev inequality (2.20) with $p = 4$ for both the velocity and the magnetic field. Indeed, we consider an $\boldsymbol{L}^4$-$\boldsymbol{L}^4$-$\boldsymbol{L}^2$ argument when applying the Hölder inequality. We note that, in the proof given in [32, Lemma 4.3], an argument $\boldsymbol{L}^3$-$\boldsymbol{L}^6$-$\boldsymbol{L}^2$ is used to handle the form C_h. This approach is realistic because they consider the general case of a bounded Lipschitz domain $\Omega \in \mathbb{R}^3$ and a Dirichlet boundary condition for the veloicty. So, in this case, the discrete inequality given in (2.19) is used for the velocity $\boldsymbol{u}$ with $r = 6$. However, this inequality can not be applied for the magnetic field $\boldsymbol{b}$. With boundary conditions of type (2.1e) or $\boldsymbol{b}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ, $\boldsymbol{b}$ has a regularity only in $\boldsymbol{H}^{1/2}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Hence, they establish in [32, Theorem 8.1] the following discrete $\boldsymbol{L}^3$ estimate for functions in $\boldsymbol{C}_h$.

Theorem 2.4 *[W. Qiu and Ke. Shi, 2020] There is a positive constant C such that for any $\boldsymbol{b}_h \in \boldsymbol{C}_h$ we have*

$$\|\boldsymbol{b}_h\|_{\boldsymbol{L}^3(\Omega)} \leqslant C\Big(\sum_{e\in\mathcal{F}_h^I}\|h^{-\frac{1}{2}}[\![\boldsymbol{b}_h]\!]_T\|^2_{0,e} + \sum_{T\in\mathcal{T}_h}\|\mathbf{curl}\,\boldsymbol{b}_h\|^2_{0,T} + \|\,\mathrm{div}_h^N\,\boldsymbol{b}_h\|_{L^2(\Omega)}\Big), \tag{2.37}$$

where the discrete divergence $\mathrm{div}_h^N\,\boldsymbol{b}_h$ is defined to be the unique function in $H^1(\Omega)\cap L^2_0(\Omega)\cap S_h$ satisfying

$$(\mathrm{div}_h^N\,\boldsymbol{b}_h, s)_{\mathcal{T}_h} = -(\boldsymbol{b}_h, \nabla s)_{\mathcal{T}_h}, \quad \textit{for all } s \in H^1(\Omega)\cap L^2_0(\Omega)\cap S_h,$$

where $S_h := \big\{s_h \in L^2(\Omega);\ s_{h|T} \in \mathcal{P}_{k+1}(T)\,,\, \forall T \in \mathcal{T}_h\big\}$.

We can now derive the continuity of the form $\boldsymbol{C}_h$ as in [32, Lemma 4.3]. The main tool different from their proof is that we use an $\boldsymbol{L}^4$-$\boldsymbol{L}^4$-$\boldsymbol{L}^2$ argument instead of an $\boldsymbol{L}^3$-$\boldsymbol{L}^6$-$\boldsymbol{L}^2$ argument when applying the Hölder inequality. So, for our case, we apply the discrete Sobolev inequality (2.20) with $p = 4$ instead of (2.19) and (2.37). The proof is given for the sake of completeness.

Lemma 2.5 *There exists a constant $C > 0$ such that for any $(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{C}_h \times \boldsymbol{X}_h \times \boldsymbol{C}_h$,*

$$C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{c}_h) \leqslant C\kappa\,\|\boldsymbol{b}_h\|_{m,h}\,\|\boldsymbol{v}_h\|_{v,h}\,\|\boldsymbol{c}_h\|_{m,h}\,. \tag{2.38}$$

Proof. By the definition of the form $\boldsymbol{C}_h$, we have

$$C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{c}_h) = \kappa\sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{v}_h\times\boldsymbol{b}_h)\cdot\mathbf{curl}\,\boldsymbol{c}_h\,d\boldsymbol{x} - \kappa\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\boldsymbol{v}_h\times\boldsymbol{b}_h\}\!\}\cdot[\![\boldsymbol{c}_h]\!]_T\,ds \tag{2.39}$$

Applying the Hölder inequality, we obtain

$$C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{c}_h) \quad\leq\quad C\kappa\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}\Big(\sum_{T\in\mathcal{T}_h}\|\,\mathbf{curl}\,\boldsymbol{c}_h\|^2_{0,T}\Big)^{1/2}$$

$$+ \quad C\kappa\Big(\sum_{e\in\mathcal{F}_h^I} h\|\boldsymbol{b}_h\|^4_{\boldsymbol{L}^4(e)}\Big)^{1/4}\Big(\sum_{e\in\mathcal{F}_h^I} h\|\boldsymbol{v}_h\|^4_{\boldsymbol{L}^4(e)}\Big)^{1/4}\Big(\sum_{e\in\mathcal{F}_h^I} h^{-1}\|[\![\boldsymbol{c}_h]\!]_T\|^2_{\boldsymbol{L}^2(e)}\Big)^{1/2}.$$

Thanks to the trace inequality, we have

$$\begin{aligned} C_h(\boldsymbol{b}_h,\boldsymbol{v}_h,\boldsymbol{c}_h) \leq C\kappa\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}\Big(\sum_{T\in\mathcal{T}_h}\|\,\mathbf{curl}\,\boldsymbol{c}_h\|^2_{0,T}\Big)^{1/2} \\ + C\kappa\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}\Big(\sum_{e\in\mathcal{F}_h^I}\frac{\sigma_1}{h}\|[\![\boldsymbol{c}_h]\!]_T\|^2_{\boldsymbol{L}^2(e)}\Big)^{1/2} \\ \leq C\kappa\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}\,\|\boldsymbol{c}_h\|_{m,h}\,. \end{aligned}$$

Then, the bound (2.38) is a direct application of the Sobolev's inequalities (2.37) and (2.20) with $p = 4$. □

Remark 2.5 *It is not possible to reuse the same strategy in [32] with a regularity below $\boldsymbol{H}^1(\Omega)$ for both the velocity and the magnetic field. Indeed, in our work, the velocity and the magnetic field verify the same type of boundary conditions. So, if we consider the case of non smooth domain, we have $\boldsymbol{u} \in \boldsymbol{H}^{1/2}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ and $\boldsymbol{b} \in \boldsymbol{H}^{1/2}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Then, the argument $\boldsymbol{L}^3$-$\boldsymbol{L}^3$-$\boldsymbol{L}^2(\mathcal{T}_h)$ does not allow to bound $\sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{v}\times\boldsymbol{b})\cdot\mathbf{curl}\,\boldsymbol{b}\,d\boldsymbol{x}$ and then the form C_h can not be bounded.*

The well-posedness of the discrete problem (2.3) requires an inf-sup condition which is an extension of the usual inf-sup condition for the Stokes problem to our Navier-type boundary conditions. It is proven in [36, Lemma 10.2.2] that the form B_h satisfies a uniform discrete inf-sup condition. More precisely, we have

Lemma 2.6 *There exists $C_B > 0$ only depending on Ω such that :*

$$\inf_{q_h\in Q_h}\sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\frac{B_h(\boldsymbol{v}_h,q_h)}{\|\boldsymbol{v}_h\|_{v,h}\,\|q_h\|_{Q_h}} \geqslant C_B, \tag{2.40}$$

2.4 Well-posedness of the discrete problem

We aim in this subsection to show the solvability of (2.3) by analysing an equivalent fixed-point problem. To this end, we first eliminate the pressure by introducing the space $\boldsymbol{K}_h$ defined by :

$$\boldsymbol{K}_h = \{(\boldsymbol{u}_h,\boldsymbol{b}_h)\in\boldsymbol{W}_h;\quad \mathcal{B}_h((\boldsymbol{u}_h,\boldsymbol{b}_h),q_h) = B_h(\boldsymbol{u}_h,q_h) = 0,\quad \forall q_h\in Q_h\}.$$

So, we consider the following problem : Find $(\boldsymbol{u}_h,\,\boldsymbol{b}_h)\in\boldsymbol{K}_h$ such that

$$\mathcal{A}_h((\boldsymbol{u}_h,\boldsymbol{b}_h),(\boldsymbol{u}_h,\boldsymbol{b}_h),(\boldsymbol{v}_h,\boldsymbol{c}_h)) = \mathcal{L}_h((\boldsymbol{v}_h,\boldsymbol{c}_h)), \tag{2.41}$$

for all $(\boldsymbol{v}_h,\,\boldsymbol{c}_h)\in\boldsymbol{K}_h$. Newt, for given $(\boldsymbol{w}_h,\boldsymbol{d}_h)\in\boldsymbol{K}_h$, we define the operator $\mathscr{F}_h$ by :

$$\begin{aligned} &\mathscr{F}_h\colon \boldsymbol{K}_h \to \boldsymbol{K}_h \\ &(\boldsymbol{w}_h,\boldsymbol{d}_h)\mapsto \mathscr{F}_h(\boldsymbol{w}_h,\boldsymbol{d}_h) = (\boldsymbol{u}_h,\boldsymbol{b}_h), \end{aligned}$$

where $(\boldsymbol{u}_h,\boldsymbol{b}_h)$ is the solution of the linearized problem : Find $(\boldsymbol{u}_h,\,\boldsymbol{b}_h)\in\boldsymbol{K}_h$ such that

$$\mathcal{A}_h((\boldsymbol{w}_h,\boldsymbol{d}_h),(\boldsymbol{u}_h,\boldsymbol{b}_h),(\boldsymbol{v}_h,\boldsymbol{c}_h)) = \mathcal{L}_h((\boldsymbol{v}_h,\boldsymbol{c}_h)), \tag{2.42}$$

for all $(\boldsymbol{v}_h,\, \boldsymbol{c}_h) \in \boldsymbol{K}_h$. Then, the discrete DG scheme (2.41) can be rewritten equivalently as the following fixed-point problem :

$$\text{Find } (\boldsymbol{u}_h,\, \boldsymbol{b}_h) \in \boldsymbol{K}_h \quad \text{such that} \quad \mathscr{F}_h(\boldsymbol{w}_h, \boldsymbol{d}_h) = (\boldsymbol{u}_h,\, \boldsymbol{b}_h). \tag{2.43}$$

In what follows, we focus on analysing the existence and uniqueness of such a fixed point $(\boldsymbol{u}_h,\, \boldsymbol{b}_h)$. For this purpose, we will verify that the operator $\mathscr{F}_h$ satisfies the hypothesis of the Brouwer's fixed-point theorem (c.f. [8, Theorem 9.9-2]) which is stated as follows : let W be a nonempty compact convex subset of a finite-dimensional normed space, and let $S : W \to W$ be a contraction from W into itself. Then S has a unique fixed point in W. The existence and uniqueness of p_h follow from the discrete inf-sup condition for the incompressibility form B_h.

We begin by checking that the operator $\mathscr{F}$ is well defined. So, in the following result, we show that the linearized problem (2.42) has a unique solution.

Lemma 2.7 *Assuming the stabilization parameters σ_1, σ_2, m_1 and m_2 sufficiently large, there exists a unique solution $(\boldsymbol{u}_h, \boldsymbol{b}_h) \in \boldsymbol{K}_h$ for the linearized problem (2.42). Moreover, we have the following estimate :*

$$\nu \left\|\boldsymbol{u}_h\right\|_{v,h}^2 + \kappa\mu \left\|\boldsymbol{b}_h\right\|_{m,h}^2 \leqslant C(\nu^{-1} \left\|\boldsymbol{f}\right\|_{0,\Omega}^2 + \kappa^{-1}\mu^{-1} \left\|\boldsymbol{g}\right\|_{0,\Omega}^2). \tag{2.44}$$

Proof. Since (2.42) consists of a linear system, it suffices to establish the uniqueness. Assume that the data are zero $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$ and we prove that $(\boldsymbol{u}_h,\, \boldsymbol{b}_h) = (\boldsymbol{0}, \boldsymbol{0})$. Choosing $(\boldsymbol{v}_h, \boldsymbol{c}_h) = (\boldsymbol{u}_h, \boldsymbol{b}_h)$ as test functions leads to

$$\nu \left\|\boldsymbol{u}_h\right\|_{v,h}^2 + \kappa\mu \left\|\boldsymbol{b}_h\right\|_{m,h}^2 = 0,$$

implying that $\boldsymbol{u}_h = \boldsymbol{b}_h = \boldsymbol{0}$. The estimate in (2.44) follows as a consequence of the coercivity of A_h and M_h stated in Lemma 2.2. Indeed, we have :

$$\mathcal{A}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{u}_h, \boldsymbol{b}_h)) = A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + M_h(\boldsymbol{b}_h, \boldsymbol{b}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_h \, d\boldsymbol{x} + \int_\Omega \boldsymbol{g} \cdot \boldsymbol{b}_h \, d\boldsymbol{x}. \tag{2.45}$$

Thanks to (2.32), (2.33) and applying Cauchy-Schwarz inequality, we derive :

$$\nu \left\|\boldsymbol{u}_h\right\|_{v,h}^2 + \kappa\mu |\boldsymbol{b}_h|_{m,h}^2 \quad \leqslant \quad C\Big(\left\|\boldsymbol{f}\right\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{u}_h\right\|_{\boldsymbol{L}^2(\Omega)} + \left\|\boldsymbol{g}\right\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{b}_h\right\|_{\boldsymbol{L}^2(\Omega)} \Big).$$

The Sobolev inequality (2.20) with $p = 2$ and Young's inequality imply the desired estimate. □

In the following result, we provide the assumptions under which problem (2.3) is well defined.

Theorem 2.6 *Assume that*

$$\nu^{-2} \left\|\boldsymbol{f}\right\|_{0,\Omega},\ \mu^{-1}\nu^{-1} \left\|\boldsymbol{f}\right\|_{0,\Omega},\ \kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}}\nu^{-\frac{3}{2}} \left\|\boldsymbol{g}\right\|_{0,\Omega},\ \nu^{-\frac{1}{2}}\mu^{-\frac{3}{2}}\kappa^{-\frac{1}{2}} \left\|\boldsymbol{g}\right\|_{0,\Omega} \tag{2.46}$$

are small enough. Then, the DG scheme (2.3) has a unique solution $(\boldsymbol{u}_h, \boldsymbol{b}_h, p_h) \in \boldsymbol{X}_h \times \boldsymbol{C}_h \times Q_h$ which satisfies :

$$\nu \left\|\boldsymbol{u}_h\right\|_{v,h}^2 + \kappa\mu \left\|\boldsymbol{b}_h\right\|_{m,h}^2 \quad \leqslant \quad C(\nu^{-1} \left\|\boldsymbol{f}\right\|_{0,\Omega}^2 + \kappa^{-1}\mu^{-1} \left\|\boldsymbol{g}\right\|_{0,\Omega}^2) \tag{2.47}$$

Proof. We begin by proving the existence and uniqueness of $(\boldsymbol{u}_h, \boldsymbol{b}_h)$ solution of (2.41). This is equivalent to showing that the operator $\mathscr{F}_h$ defined in (2.43) has a unique fixed-point. Two steps are needed to that.

Step 1 : We prove that $\mathcal{F}_h$ maps a closed ball into itself. Using the same arguments as above, we can show that if $(\boldsymbol{u}_h,\, \boldsymbol{b}_h)$ is a solution of (2.41), then the following *a priori* estimate holds

$$\nu \left\|\boldsymbol{u}_h\right\|_{v,h}^2 + \kappa\mu \left\|\boldsymbol{b}_h\right\|_{m,h}^2 \leqslant C\Big(\nu^{-1} \left\|\boldsymbol{f}\right\|_{\boldsymbol{L}^2(\Omega)}^2 + \kappa^{-1}\mu^{-1} \left\|\boldsymbol{g}\right\|_{\boldsymbol{L}^2(\Omega)}^2\Big) := M \tag{2.48}$$

So, we define $\boldsymbol{G}_M$ as :

$$\boldsymbol{G}_M := \left\{(\boldsymbol{u}_h, \boldsymbol{b}_h) \in \boldsymbol{K}_h,\ \nu \|\boldsymbol{u}_h\|^2_{v,h} + \kappa\mu \|\boldsymbol{b}_h\|^2_{m,h} \leqslant M\right\}. \tag{2.49}$$

It is easy to see that $\boldsymbol{G}_M$ is a closed set of $\boldsymbol{K}_h$ and $\mathscr{F}(\boldsymbol{G}_M) \subset \boldsymbol{G}_M$.

Step 2 : We prove that when the data $\boldsymbol{f}$ and $\boldsymbol{g}$ are small enough, $\mathscr{F}_h$ is a contraction from $\boldsymbol{G}_M$ into itself. We remark in advance that a combination of the Cauchy-Schwarz and Hölder inequalities with the discrete Sobolev's inequality (2.20) plays a key role in the proof.

Let $(\boldsymbol{w}_h^1, \boldsymbol{d}_h^1), (\boldsymbol{w}_h^2, \boldsymbol{d}_h^2) \in \boldsymbol{G}_M$ and $(\boldsymbol{u}_h^1, \boldsymbol{b}_h^1) := \mathcal{F}(\boldsymbol{w}_h^1, \boldsymbol{d}_h^1)$, $(\boldsymbol{u}_h^2, \boldsymbol{b}_h^2) := \mathcal{F}(\boldsymbol{w}_h^2, \boldsymbol{d}_h^2)$ be the solutions of the linearized problems (2.42). From this fact, the differences $\boldsymbol{u}_h := \boldsymbol{u}_h^1 - \boldsymbol{u}_h^2$ and $\boldsymbol{b}_h := \boldsymbol{b}_h^1 - \boldsymbol{b}_h^2$ satisfy : for any $(\boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{W}_h$

$$\begin{aligned} & A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + M_h(\boldsymbol{b}_h, \boldsymbol{c}_h) + O_h(\boldsymbol{w}_h^1, \boldsymbol{u}_h^1, \boldsymbol{v}_h) - O_h(\boldsymbol{w}_h^2, \boldsymbol{u}_h^2, \boldsymbol{v}_h) \\ + \ & C_h(\boldsymbol{d}_h^1, \boldsymbol{v}_h, \boldsymbol{b}_h^1) - C_h(\boldsymbol{d}_h^2, \boldsymbol{v}_h, \boldsymbol{b}_h^2) - C_h(\boldsymbol{d}_h^1, \boldsymbol{u}_h^1, \boldsymbol{c}_h) + C_h(\boldsymbol{d}_h^2, \boldsymbol{u}_h^2, \boldsymbol{c}_h) = 0. \end{aligned}$$

Taking $(\boldsymbol{u}_h, \boldsymbol{b}_h)$ as test function in the above relation, adding and substracting $C_h(\boldsymbol{d}_h^1, \boldsymbol{u}_h, \boldsymbol{b}_h^2)$, we obtain :

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + M_h(\boldsymbol{b}_h, \boldsymbol{b}_h) = O_h(\boldsymbol{w}_2 - \boldsymbol{w}_1, \boldsymbol{u}_h^2, \boldsymbol{u}_h) - C_h(\boldsymbol{d}_1 - \boldsymbol{d}_2, \boldsymbol{u}_h, \boldsymbol{b}_h^2) + C_h(\boldsymbol{d}_1 - \boldsymbol{d}_2, \boldsymbol{u}_h^2, \boldsymbol{b}_h), \tag{2.50}$$

where we have used the fact that

$$O_h(\boldsymbol{w}_h^2, \boldsymbol{u}_h^2, \boldsymbol{u}_h) - O_h(\boldsymbol{w}_h^1, \boldsymbol{u}_h^1, \boldsymbol{u}_h) = O_h(\boldsymbol{w}_h^2 - \boldsymbol{w}_h^1, \boldsymbol{u}_h^2, \boldsymbol{u}_h).$$

Let us estimate each term in the right-hand side of (2.50). The first term can be easily estimated by using Lemma 2.4 and Young's inequality. Indeed, it follows that

$$\begin{aligned} O_h(\boldsymbol{w}_h^2 - \boldsymbol{w}_h^1, \boldsymbol{u}_h^2, \boldsymbol{u}_h) &\leqslant C \left\|\boldsymbol{w}_h^1 - \boldsymbol{w}_h^2\right\|_{v,h} \left\|\boldsymbol{u}_h^2\right\|_{v,h} \|\boldsymbol{u}_h\|_{v,h} \\ &\leq C\nu^{-1} \left\|\boldsymbol{w}_h^1 - \boldsymbol{w}_h^2\right\|^2_{v,h} \left\|\boldsymbol{u}_h^2\right\|^2_{v,h} + \frac{1}{4}\nu \|\boldsymbol{u}_h\|^2_{v,h}. \end{aligned}$$

Next, by using Corollary 2.5, we have

$$\begin{aligned} C_h(\boldsymbol{d}_h^1 - \boldsymbol{d}_h^2, \boldsymbol{u}_h, \boldsymbol{b}_h^2) &\leqslant C\kappa \left\|\boldsymbol{d}_h^1 - \boldsymbol{d}_h^2\right\|_{m,h} \|\boldsymbol{u}_h\|_{v,h} \left\|\boldsymbol{b}_h^2\right\|_{m,h} \\ &\leq C\kappa^2\nu^{-1} \left\|\boldsymbol{d}_h^1 - \boldsymbol{d}_h^2\right\|^2_{m,h} \left\|\boldsymbol{b}_h^2\right\|^2_{m,h} + \frac{1}{4}\nu \|\boldsymbol{u}_h\|^2_{v,h}, \end{aligned}$$

$$\begin{aligned} C_h(\boldsymbol{d}_h^1 - \boldsymbol{d}_h^2, \boldsymbol{u}_h^2, \boldsymbol{b}_h) &\leqslant C\kappa \left\|\boldsymbol{d}_h^1 - \boldsymbol{d}_h^2\right\|_{m,h} \left\|\boldsymbol{u}_h^2\right\|_{v,h} \|\boldsymbol{b}_h\|_{m,h} \\ &\leq C\kappa\mu^{-1} \left\|\boldsymbol{d}_h^1 - \boldsymbol{d}_h^2\right\|^2_{m,h} \left\|\boldsymbol{u}_h^2\right\|^2_{v,h} + \frac{1}{2}\kappa\mu \|\boldsymbol{b}_h\|^2_{m,h}. \end{aligned}$$

Collecting the above estimates in (2.50), using the coercivity of A_h and M_h, we obtain

$$\begin{aligned} \nu \|\boldsymbol{u}_h\|^2_{v,h} + \kappa\mu \|\boldsymbol{b}_h\|^2_{m,h} &\leqslant C\nu \left\|\boldsymbol{w}_h^1 - \boldsymbol{w}_h^2\right\|^2_{m,h} \nu^{-2} \left\|\boldsymbol{u}_h^2\right\|^2_{v,h} \\ &+ C\big(\kappa\mu^{-1}\nu^{-1} \left\|\boldsymbol{b}_h^2\right\|^2_{m,h} + \kappa\mu^{-2} \left\|\boldsymbol{u}_h^2\right\|^2_{v,h}\big)\kappa\mu \left\|\boldsymbol{d}_h^1 - \boldsymbol{d}_h^2\right\|^2_{m,h} \end{aligned}$$

So, if $\nu^{-2} \left\|\boldsymbol{u}_h^2\right\|_{1,h} < 1$ and $\kappa\mu^{-1}\nu^{-1} \left\|\boldsymbol{b}_h^2\right\|^2_C + \kappa\mu^{-2} \left\|\boldsymbol{u}_h^2\right\|^2_{1,h} < 1$, that is, if the smallness conditions (2.46) on $\boldsymbol{f}$ and $\boldsymbol{g}$ are satisfied, the operator $\mathscr{F}_h$ is a contraction on $\boldsymbol{G}_M$.

The above proofs show that $\mathscr{F}_h$ satisfies the hypotheses of Brouwer's fixed-point theorem on $\boldsymbol{G}_M$. Then, $\mathcal{F}_h$ has a unique fixed-point $(\boldsymbol{u}_h,\ \boldsymbol{b}_h)$ in $\boldsymbol{G}_M$. Therefore, the existence and uniqueness of the solution $(\boldsymbol{u}_h, \boldsymbol{b}_h)$ has been proved. In turn, the a priori estimate (2.47) follows directly from (2.48). Now that $\boldsymbol{u}_h$ and $\boldsymbol{b}_h$ have been computed, we want to recover the pressure p_h. Observe that p_h is the solution of

$$B_h(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h\,d\boldsymbol{x} - A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) - O_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) - C_h(\boldsymbol{b}_h,\boldsymbol{v}_h,\boldsymbol{b}_h) \tag{2.51}$$

Applying Lemma 2.1, Lemma 2.3, Lemma 2.4 and Corollary 2.5 to bound A_h, O_h and C_h respectively, we deduce that the right-hand side of the above equation defines a continuous linear functional on $\boldsymbol{X}_h$. The inf-sup condition in Lemma 2.6 gives the existence and uniqueness of $p_h \in Q_h$ to the problem (2.51). □

3 Error analysis

In this subsection, we present a priori error bounds for the proposed DG method. The proof uses essentially the same techniques as in [32] where a similar result has been proved for the MHD problem with Dirichlet boundary condition for the velocity together with zero tangential trace for the magnetic field. We note that the solution $(\boldsymbol{u},\boldsymbol{b},p)$ of the continuous MHD problem (2.1) satisfies

$$\mathcal{A}_h((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v}_h,\boldsymbol{c}_h)) + \mathcal{B}_h((\boldsymbol{v}_h,\boldsymbol{c}_h),p) = \mathcal{L}((\boldsymbol{v}_h,\boldsymbol{c}_h)), \quad \forall(\boldsymbol{v}_h,\boldsymbol{c}_h)\in\boldsymbol{W}_h, \tag{3.1a}$$

$$\mathcal{B}_h((\boldsymbol{u},\boldsymbol{b}),q_h) = 0, \quad \forall q_h \in Q_h. \tag{3.1b}$$

Let us begin by introducing an approximation result for the space $\boldsymbol{X}_h$ (see [19]). For $k = 1,2,3,$ there exists a continuous interpolation operator I_h defined from $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{X}_h$ such that, for all $T\in\mathcal{T}_h$ and $e\in\mathcal{F}_h$:

$$\forall \boldsymbol{v}\in\boldsymbol{H}^1(\Omega),\ \ \forall q_h\in\mathcal{P}_{k-1}(T),\ \int_T q_h\,\mathrm{div}(I_h(\boldsymbol{v})-\boldsymbol{v})\,dx = 0, \tag{3.2a}$$

$$\forall \boldsymbol{v}\in\boldsymbol{H}^1_T(\Omega),\ \forall e\in\mathcal{F}_h,\ \ \forall q_h\in\mathcal{P}_{k-1}(e),\ \ \int_e q_h\,[\![I_h(\boldsymbol{v})]\!]\,ds = 0. \tag{3.2b}$$

Moreover, for $s\in[1,k+1]$ the following interpolation estimate holds :

$$\forall \boldsymbol{v}\in\boldsymbol{H}^s(\Omega), \quad \|I_h(\boldsymbol{v})-\boldsymbol{v}\|_{1,T} \leqslant C h_T^{s-1}\|\boldsymbol{v}\|_{s,\Delta_T}, \tag{3.3}$$

where Δ_T is a suitable macro-element containing T.

We use the L^2-projection of degree $k-1$ onto Q_h to approximate the pressure p. So there exists approximationz $\Pi_Q\in Q_h$ (see [19]), defined on each $T\in\mathcal{T}_h$ by

$$\forall q\in\mathcal{P}_{k-1}(T), \quad \int_T q(p-\Pi_Q p)\,d\boldsymbol{x} = 0, \tag{3.4}$$

and satisy the following approximations properties for every integer $s\in[0,k]$:

$$\|p-\Pi_Q p\|_{0,T} \le C h_T^s \|p\|_{s,T}, \qquad \forall p\in H^s(\Omega)\cup L^2_0(\Omega). \tag{3.5}$$

We split the errors in two parts : $\boldsymbol{e}^u = \boldsymbol{u}-\boldsymbol{u}_h$, $\boldsymbol{e}^b = \boldsymbol{b}-\boldsymbol{b}_h$ and $e^p = p - p_h$ with

$$\begin{aligned}
\boldsymbol{\chi}_u &= I_h(\boldsymbol{u}) - \boldsymbol{u}_h, & \boldsymbol{\eta}_u &= \boldsymbol{u} - I_h(\boldsymbol{u})\\
\boldsymbol{\chi}_b &= I_h(\boldsymbol{b}) - \boldsymbol{b}_h, & \boldsymbol{\eta}_b &= \boldsymbol{b} - I_h(\boldsymbol{b})\\
\chi_p &= \Pi_Q(p) - p_h, & \eta_p &= p - \Pi_Q(p)
\end{aligned}$$

We have the following lemma for the error projection :

Lemma 3.1 *Let $(\boldsymbol{u}, \boldsymbol{b}, p) \in \boldsymbol{H}^{k+1}(\Omega) \times \boldsymbol{H}^{k+1}(\Omega) \times H^k(\Omega)$ be a ssolution of the continuous MHD problem (2.1). Let $(\boldsymbol{u}_h, p_h, \boldsymbol{b}_h)$ be the solution in $\boldsymbol{X}_h \times Q_h \times \boldsymbol{C}_h$ of the DG scheme (2.3). In addition to the assumptions in Theorem 2.6, we assume that $\frac{1}{\min(\nu,\mu)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and $\frac{\kappa^{1/2}}{\nu^{1/2}\mu^{1/2}} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}$ are small enough. Then, we have the following estimate on the errors :*

$$\begin{aligned} &\nu \|\boldsymbol{\chi}_u\|_{v,h}^2 + \kappa\mu \|\boldsymbol{\chi}_b\|_{m,h}^2 \\ &\leqslant Ch^{2k}\Big(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 + \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 + \|p\|_{H^k(\Omega)}^2 + \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 \big(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 + \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 \big)\Big) \end{aligned} \tag{3.6}$$

with C depending on the datas of the problem but not of the mesh size h.

Proof. From (3.1), we have for all $(\boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$:

$$\begin{aligned} &A_h(\boldsymbol{u} - I_h(\boldsymbol{u}), \boldsymbol{v}_h) + A_h(I_h(\boldsymbol{u}), \boldsymbol{v}_h) + M_h(\boldsymbol{b} - I_h(\boldsymbol{b}), \boldsymbol{c}_h) + M_h(I_h(\boldsymbol{b}), \boldsymbol{c}_h) + O_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) \\ &+ C_h(\boldsymbol{b}, \boldsymbol{v}_h, \boldsymbol{b}) - C_h(\boldsymbol{b}, \boldsymbol{u}, \boldsymbol{c}_h) + B_h(\boldsymbol{v}_h, p - \Pi_Q(p)) + B_h(\boldsymbol{v}_h, \Pi_Q(p)) = \mathcal{L}(\boldsymbol{v}_h, \boldsymbol{c}_h), \\ &B_h(\boldsymbol{u} - I_h(\boldsymbol{u}), q_h) + B_h(I_h(\boldsymbol{u}), q_h) = 0. \end{aligned} \tag{3.7}$$

By the definition of I_h (c.f. (3.2a) and (3.2b)) and Π_Q (c.f. (3.4)), we have :

$$\begin{aligned} B_h(\boldsymbol{v}_h, \Pi_Q(p) - p) &= -\sum_{T\in\mathcal{T}_h} \int_T (\operatorname{div} \boldsymbol{v}_h)(\Pi_Q(p) - p)\, dx + \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\Pi_Q(p) - p\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds \\ &= \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\Pi_Q(p) - p\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds \\ B_h(\boldsymbol{u} - I_h(\boldsymbol{u}), q_h) &= -\sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(I_h(\boldsymbol{u}) - \boldsymbol{u}) q_h \, dx + \sum_{e\in\mathcal{F}_h} \int_e \{\!\{q_h\}\!\} [\![I_h(\boldsymbol{u}) - \boldsymbol{u}]\!]_N \, ds \\ &= 0. \end{aligned}$$

So, from (3.7), we obtain :

$$\begin{aligned} &A_h(I_h(\boldsymbol{u}), \boldsymbol{v}_h) + M_h(I_h(\boldsymbol{b}), \boldsymbol{c}_h) + O_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) + C_h(\boldsymbol{b}, \boldsymbol{v}_h, \boldsymbol{b}) - C_h(\boldsymbol{b}, \boldsymbol{u}, \boldsymbol{c}_h) + B_h(\boldsymbol{v}_h, \Pi_Q(p)) \\ &= A_h(I_h(\boldsymbol{u}) - \boldsymbol{u}, \boldsymbol{v}_h) + M_h(I_h(\boldsymbol{b}) - \boldsymbol{b}, \boldsymbol{c}_h) + \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\Pi_Q(p) - p\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds + \mathcal{L}(\boldsymbol{v}_h, \boldsymbol{c}_h), \\ &B_h(I_h(\boldsymbol{u}), q_h) = 0. \end{aligned}$$

We next substract from (2.3) to obtain

$$\begin{aligned} &A_h(\boldsymbol{\chi}_u, \boldsymbol{v}_h) + M_h(\boldsymbol{\chi}_b, \boldsymbol{c}_h) + B_h(\boldsymbol{v}_h, \chi_p) \\ &= O_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - O_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) + C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{b}_h) - C_h(\boldsymbol{b}, \boldsymbol{v}_h, \boldsymbol{b}) + C_h(\boldsymbol{b}, \boldsymbol{u}, \boldsymbol{c}_h) - C_h(\boldsymbol{b}_h, \boldsymbol{u}_h, \boldsymbol{c}_h) \\ &+ A_h(\underbrace{I_h(\boldsymbol{u}) - \boldsymbol{u}}_{-\boldsymbol{\eta}_u}, \boldsymbol{v}_h) + M_h(\underbrace{I_h(\boldsymbol{b}) - \boldsymbol{b}}_{-\boldsymbol{\eta}_b}, \boldsymbol{c}_h) + \sum_{e\in\mathcal{F}_h} \int_e \underbrace{\{\!\{\Pi_Q(p) - p\}\!\}}_{-\{\!\{\eta_p\}\!\}} [\![\boldsymbol{v}_h]\!]_N \, ds, \\ &B_h(\boldsymbol{\chi}_u, q_h) = 0. \end{aligned} \tag{3.8}$$

Combining inequalities of (3.8) with test functions $(\boldsymbol{\chi}_u, \boldsymbol{\chi}_b, \chi_p)$ and using (2.32)-(2.33), we obtain :

$$\begin{aligned} C\big(\nu \|\boldsymbol{\chi}_u\|_{v,h}^2 + \kappa\mu \|\boldsymbol{\chi}_b\|_{m,h}^2\big) &\leq A_h(\boldsymbol{\chi}_u, \boldsymbol{\chi}_u) + M_h(\boldsymbol{\chi}_b, \boldsymbol{\chi}_b) \\ &= -A_h(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) - M_h(\boldsymbol{\eta}_b, \boldsymbol{\chi}_b) - \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\eta_p\}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds \\ &- \big(O_h(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\chi}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\eta}_u, \boldsymbol{u}, \boldsymbol{\chi}_u)\big) \\ &+ \big(C_h(\boldsymbol{\eta}_b, \boldsymbol{u}, \boldsymbol{\chi}_b) - C_h(\boldsymbol{\eta}_b, \boldsymbol{\chi}_u, \boldsymbol{b})\big) - \big(C_h(\boldsymbol{\chi}_b, \boldsymbol{\chi}_u, \boldsymbol{b}) - C_h(\boldsymbol{\chi}_b, \boldsymbol{u}, \boldsymbol{\chi}_b)\big) \\ &+ \big(C_h(\boldsymbol{b}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_b) - C_h(\boldsymbol{b}_h, \boldsymbol{\chi}_u, \boldsymbol{\eta}_b)\big). \end{aligned} \tag{3.9}$$

Let us bound every term in the right hand side of (3.9).
By the definition of A_h, we have

$$\begin{aligned} A_h(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) := \nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{\eta}_u \cdot \mathbf{curl}\ \boldsymbol{\chi}_u \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T (\mathrm{div}\, \boldsymbol{\eta}_u)(\mathrm{div}\, \boldsymbol{\chi}_u) \, d\boldsymbol{x} \\ - \nu \sum_{e\in\mathcal{F}_h^I} \Big(\int_e \{\!\{\mathbf{curl}\ \boldsymbol{\eta}_u\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{\chi}_u\}\!\} \cdot [\![\boldsymbol{\eta}_u]\!]_T \, ds \Big) \\ - \nu \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{\mathrm{div}\ \boldsymbol{\eta}_u\}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds + \int_e \{\!\{\mathrm{div}\, \boldsymbol{\chi}_u\}\!\} [\![\boldsymbol{\eta}_u]\!]_N \, ds \Big) \\ \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{\eta}_u]\!]_T \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{\eta}_u]\!]_N [\![\boldsymbol{\chi}_u]\!]_N \, ds \end{aligned} \tag{3.10}$$

Applying Cauchy-Schwarz inequality and (3.3), we have for the terms in the first row of (3.10) :

$$\begin{aligned} &\nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{\eta}_u \cdot \mathbf{curl}\ \boldsymbol{\chi}_u \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T (\mathrm{div}\, \boldsymbol{\eta}_u)(\mathrm{div}\, \boldsymbol{\chi}_u) \, d\boldsymbol{x} \\ &\leq \nu \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{\eta}_u\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\mathrm{div}\, \boldsymbol{\eta}_u\|_{0,T}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{\chi}_u\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\mathrm{div}\, \boldsymbol{\chi}_u\|_{0,T}^2 \Big)^{1/2} \\ &\leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} \end{aligned}$$

Moreover, combining the continuity of $\boldsymbol{u}$ and (3.2b), we have

$$\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{\chi}_u\}\!\} \cdot [\![\boldsymbol{\eta}_u]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\mathrm{div}\, \boldsymbol{\chi}_u\}\!\} [\![\boldsymbol{\eta}_u]\!]_N \, ds = 0.$$

So, for the second and the third rows of (3.10), it remains to bound the terms :

$$\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\ \boldsymbol{\eta}_u\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \quad \text{and} \quad \nu \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\mathrm{div}\ \boldsymbol{\eta}_u\}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds$$

Let us give detail proof for the bound of the first term. For this, let us introduce the standard Lagrange interpolation operator of polynomial degree k, denoted Π_h. So, we have

$$\int_e \{\!\{\mathbf{curl}\ \boldsymbol{\eta}_u\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds = \int_e \big(\{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T + \{\!\{\mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \big) \, ds \tag{3.11}$$

For the first term in (3.11), we have

$$\int_e \{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \leq \|\{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\}\|_{0,e} \|[\![\boldsymbol{\chi}_u]\!]_T\|_{0,e}$$

Let $e \in \mathcal{F}_h^I$ such that $e = \partial T_1 \cap \partial T_2$ with $T_1, T_2 \in \mathcal{T}_h$. Using the discrete trace inequality and the approximation property of the Lagrange interpolation, we obtain

$$\begin{aligned} \frac{1}{\sqrt{|e|}} \|\{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\}\|_{0,e} \leq C \Big(\frac{1}{h_{T_1}} \|\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{0,T_1} + \frac{1}{h_{T_2}} \|\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{0,T_2} \\ + |\, \mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})|_{1,T_1\cup T_2} \Big) \leq C h^{k-1} \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(T_1\cup T_2)} \end{aligned}$$

So, we have

$$\begin{aligned} \nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \Big(\sum_{e\in\mathcal{F}_h^I} \frac{1}{|e|} \|[\![\boldsymbol{\chi}_u]\!]_T\| \Big)^{1/2} \\ \leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} \end{aligned} \tag{3.12}$$

For the second term in (3.11), since $\Pi_h \boldsymbol{u} - I_h \boldsymbol{u}$ is polynomial, we can write

$$\frac{1}{\sqrt{|e|}} \|\{\!\{\mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u})\}\!\}\|_{0,e} \leq C\Big(\frac{1}{h_{T_1}} \|\mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u})\|_{0,T_1} + \frac{1}{h_{T_2}} \|\mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u})\|_{0,T_2}\Big)$$

By triangle inequality and (3.3), we have for each element T of $\mathcal{T}_h$

$$\|\mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u})\|_{0,T} \leq \|\mathbf{curl}(\Pi_h \boldsymbol{u} - \boldsymbol{u})\|_{0,T} + \|\mathbf{curl}(\boldsymbol{u} - I_h \boldsymbol{u})\|_{0,T} \leq Ch_T^k \|\boldsymbol{u}\|_{k+1,\Delta_T}$$

Then, we have

$$\nu \sum_{e\in\mathcal{F}_h^I} \int_e \big(\{\!\{\mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\ \boldsymbol{\eta}_u\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} . \tag{3.14}$$

Similarly, we can obtain

$$\nu \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\mathrm{div}\ \boldsymbol{\eta}_u\}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds \leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} . \tag{3.15}$$

For the terms in the last row of (3.10), we use (3.3) and similar arguments as above to obtain

$$\nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{\eta}_u]\!]_T \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} , \tag{3.16}$$

$$\nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{\eta}_u]\!]_N [\![\boldsymbol{\chi}_u]\!]_N \, ds \leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} , \tag{3.17}$$

Collecting the bounds (3.11), (3.11)-(3.17) yields

$$A_h(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) \leq C\nu h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} \leq \frac{\nu}{10} \|\boldsymbol{\chi}_u\|_{v,h}^2 + Ch^{2k} \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 . \tag{3.18}$$

Next, in view of the definition of M_h and the fact that $\boldsymbol{C}_h = \boldsymbol{X}_h$, the terms of M_h can be bounded in a similar way as the terms of A_h to obtain :

$$M_h(\boldsymbol{\eta}_b, \boldsymbol{\chi}_b) \leqslant C\kappa\mu h^k \|\boldsymbol{\chi}_b\|_{m,h} \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)} \leq \frac{\kappa\mu}{8} \|\boldsymbol{\chi}_b\|_{m,h}^2 + Ch^{2k} \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 . \tag{3.19}$$

Similarly, using the estimate (3.5) for the L^2-projection Π_Q, we obtain :

$$\begin{aligned}\sum_{e\in\mathcal{F}_h} \int_e \{\!\{\eta_p\}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds &\leqslant C \sum_{e\in\mathcal{F}_h} \Big(\int_e h_e \{\!\{\eta_p\}\!\}^2 \, ds\Big)^{\frac{1}{2}} \Big(\int_e h_e^{-1} [\![\boldsymbol{\chi}_u]\!]_N{}^2 \, ds\Big)^{\frac{1}{2}} \\ &\leqslant Ch^k \|\boldsymbol{\chi}_u\|_{v,h} \|p\|_{H^k(\Omega)} \leq \frac{\nu}{10} \|\boldsymbol{\chi}_u\|_{v,h}^2 + Ch^{2k} \|p\|_{H^k(\Omega)}^2 .\end{aligned} \tag{3.20}$$

For the terms on O_h in the third row of (3.9), we use Hölder's inequality to obtain :

$$\begin{aligned}&O_h(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\chi}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\eta}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) \\ &\leqslant \Big(\sum_{T\in\mathcal{F}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2\Big)^{1/2} \|\boldsymbol{\eta}_u\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{\chi}_u\|_{\boldsymbol{L}^4(\Omega)} + \Big(\sum_{T\in\mathcal{F}_h} \|\mathbf{curl}\, \boldsymbol{\chi}_u\|_{0,T}^2\Big)^{1/2} \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{\chi}_u\|_{\boldsymbol{L}^4(\Omega)} \\ &+ \Big(\sum_{T\in\mathcal{F}_h} \|\mathbf{curl}\, \boldsymbol{\eta}_u\|_{0,T}^2\Big)^{1/2} \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{\chi}_u\|_{\boldsymbol{L}^4(\Omega)}\end{aligned}$$

Using the Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$ together with the L^p discrete Sobolev's inequality (2.20) with $p = 4$ and (3.3), we obtain

$$O_h(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\chi}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\eta}_u, \boldsymbol{u}, \boldsymbol{\chi}_u)$$
$$\leqslant Ch^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \left(\|\boldsymbol{u}_h\|_{v,h} + \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \right) \|\boldsymbol{\chi}_u\|_{v,h} + C \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h}^2$$

As a consequence, we have

$$\begin{aligned} &O_h(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\chi}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\eta}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) \\ &\leqslant \frac{\nu}{10} \|\boldsymbol{\chi}_u\|_{v,h}^2 + Ch^{2k} \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 + C \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h}^2 \end{aligned} \tag{3.21}$$

For the two first terms in (3.9), we have since $[\![\boldsymbol{b}]\!]_T = \boldsymbol{0}$ on $\mathcal{F}_h^I$:

$$\begin{aligned} &C_h(\boldsymbol{\eta}_b, \boldsymbol{u}, \boldsymbol{\chi}_b) - C_h(\boldsymbol{\eta}_b, \boldsymbol{\chi}_u, \boldsymbol{b}) \\ &= \kappa \sum_{T \in \mathcal{T}_h} \left(\int_T (\boldsymbol{u} \times \boldsymbol{\eta}_b) \cdot \mathbf{curl}\, \boldsymbol{\chi}_b \, d\boldsymbol{x} - \int_T (\boldsymbol{\chi}_u \times \boldsymbol{\eta}_b) \cdot \mathbf{curl}\, \boldsymbol{b} \, d\boldsymbol{x} \right) - \kappa \sum_{e \in \mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{u} \times \boldsymbol{\eta}_b\}\!\} \cdot [\![\boldsymbol{\chi}_b]\!]_T ds \end{aligned} \tag{3.22}$$

Using (2.20) and the fact that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \quad \text{and} \quad \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^4(\Omega)} \leq C \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)},$$

the two first terms in (3.22) can be bounded as follows :

$$\begin{aligned} &\kappa \sum_{T \in \mathcal{T}_h} \left(\int_T (\boldsymbol{u} \times \boldsymbol{\eta}_b) \cdot \mathbf{curl}\, \boldsymbol{\chi}_b \, d\boldsymbol{x} - \int_T (\boldsymbol{\chi}_u \times \boldsymbol{\eta}_b) \cdot \mathbf{curl}\, \boldsymbol{b} \, d\boldsymbol{x} \right) \\ &\leqslant C\kappa \Big(\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} \|\boldsymbol{\eta}_b\|_{\boldsymbol{L}^2(\Omega)} \|\mathbf{curl}\, \boldsymbol{\chi}_b\|_{\boldsymbol{L}^2(\mathcal{T}_h)} + \|\boldsymbol{\chi}_u\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{\eta}_b\|_{\boldsymbol{L}^2(\Omega)} \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^4(\Omega)} \Big) \\ &\leqslant C\kappa \Big(h^k \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_b\|_{m,h} + h^k \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)} \Big) \\ &\leq C\kappa h^k \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)} \left(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_b\|_{m,h} + \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} \right) \end{aligned} \tag{3.23}$$

For the last term in (3.22), by using the discrete inequality and the Lagrange interpolation Π_h, we obtain :

$$\begin{aligned} &\kappa \sum_{e \in \mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{u} \times \boldsymbol{\eta}_b\}\!\} \cdot [\![\boldsymbol{\chi}_b]\!]_T ds \\ &= \kappa \sum_{e \in \mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{u} \times (I_h(\boldsymbol{b}) - \Pi_h(\boldsymbol{b}))\}\!\} \cdot [\![\boldsymbol{\chi}_b]\!]_T \, ds + \kappa \sum_{e \in \mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{u} \times (\Pi_h(\boldsymbol{b}) - \boldsymbol{b})\}\!\} \cdot [\![\boldsymbol{\chi}_b]\!]_T \, ds \\ &\leq C\kappa \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} \left(\sum_{e \in \mathcal{F}_h^I} h_e \|(I_h(\boldsymbol{b}) - \Pi_h(\boldsymbol{b}))\|_{0,e}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{F}_h^I} h_e^{-1} \|[\![\boldsymbol{\chi}_b]\!]_T\|_{0,e}^2 \right)^{\frac{1}{2}} \\ &+ C\kappa \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} \left(\sum_{e \in \mathcal{F}_h^I} h_e \|(\Pi_h(\boldsymbol{b}) - \boldsymbol{b})\|_{0,e}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{F}_h^I} h_e^{-1} \|[\![\boldsymbol{\chi}_b]\!]_T\|_{0,e}^2 \right)^{\frac{1}{2}} \\ &\leq C\kappa h^k \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_b\|_{m,h} \end{aligned} \tag{3.24}$$

Collecting (3.23) and (3.24) in (3.22), we obtain

$$\begin{aligned} C_h(\boldsymbol{\eta}_b, \boldsymbol{u}, \boldsymbol{\chi}_b) - C_h(\boldsymbol{\eta}_b, \boldsymbol{\chi}_u, \boldsymbol{b}) &\leq \kappa h^k \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)} \left(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_b\|_{m,h} + \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} \right) \\ &\leq \frac{\nu}{10} \|\boldsymbol{\chi}_u\|_{v,h}^2 + \frac{\kappa\mu}{8} \|\boldsymbol{\chi}_b\|_{m,h}^2 + Ch^{2k} \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 \end{aligned}$$

$$+ \quad Ch^{2k} \|\boldsymbol{b}\|^4_{\boldsymbol{H}^{k+1}(\Omega)} . \tag{3.25}$$

In a similar way, we have

$$\begin{aligned}
&C_h(\boldsymbol{\chi}_b, \boldsymbol{u}, \boldsymbol{\chi}_b) - C_h(\boldsymbol{\chi}_b, \boldsymbol{\chi}_u, \boldsymbol{b}) \\
&= \kappa\Big(\sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{u}\times\boldsymbol{\chi}_b)\cdot \mathbf{curl}\,\boldsymbol{\chi}_b\, dx - \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{u}\times\boldsymbol{\chi}_b\}\!\} \cdot [\![\boldsymbol{\chi}_b]\!]_T\, ds - \sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{\chi}_u\times\boldsymbol{\chi}_b)\cdot \mathbf{curl}\,\boldsymbol{b}\, dx\Big) \\
&\leqslant C\kappa \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{\chi}_b\|_{\boldsymbol{L}^4(\Omega)} \|\mathbf{curl}\,\boldsymbol{\chi}_b\|_{\boldsymbol{L}^2(\mathcal{T}_h)} \\
&+ \kappa\Big(\sum_{e\in\mathcal{F}_h^I} h\|\boldsymbol{u}\|^4_{\boldsymbol{L}^4(e)}\Big)^{1/4} \Big(\sum_{e\in\mathcal{F}_h^I} h\|\boldsymbol{\chi}_b\|^4_{\boldsymbol{L}^4(e)}\Big)^{1/4} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \|[\![\boldsymbol{\chi}_b]\!]_T\|^2_{0,e}\Big)^{\frac12} \\
&+ \kappa \|\boldsymbol{\chi}_u\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{\chi}_b\|_{\boldsymbol{L}^4(\Omega)} \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)} .
\end{aligned}$$

Then, applying (2.20) and discrete trace inequality, we have

$$\begin{aligned}
C_h(\boldsymbol{\chi}_b, \boldsymbol{u}, \boldsymbol{\chi}_b) - C_h(\boldsymbol{\chi}_b, \boldsymbol{\chi}_u, \boldsymbol{b}) &\leq C\kappa\Big(\|\boldsymbol{\chi}_b\|^2_{m,h} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h} \|\boldsymbol{\chi}_b\|_{m,h}\Big) \\
&\leq C\kappa \|\boldsymbol{\chi}_b\|^2_{m,h} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \frac{\kappa\mu}{8} \|\boldsymbol{\chi}_b\|^2_{m,h} + C\frac{\kappa}{\mu} \|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{\chi}_u\|^2_{v,h}
\end{aligned} \tag{3.26}$$

Finally, the following bound for the two last terms in (3.9) can be obtained is a similar way :

$$\begin{aligned}
&C_h(\boldsymbol{b}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_b) - C_h(\boldsymbol{b}_h, \boldsymbol{\chi}_u, \boldsymbol{\eta}_b) \\
&\leqslant C\kappa \|\boldsymbol{b}_h\|_{m,h} \Big(h^k \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \|\boldsymbol{\chi}_b\|_{m,h} + h^k \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)} \|\boldsymbol{\chi}_u\|_{v,h}\Big) \\
&\leq \frac{\kappa\mu}{8} \|\boldsymbol{\chi}_b\|^2_{m,h} + \frac{\nu}{10} \|\boldsymbol{\chi}_u\|^2_{v,h} + h^{2k} \|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} + h^{2k} \|\boldsymbol{u}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} .
\end{aligned} \tag{3.27}$$

Thus, combining all estimates, we obtain the estimate (3.6) by assuming that $\frac{1}{\min(\nu,\nu_m)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and $\frac{\sqrt{\kappa}}{\sqrt{\nu\nu_m}} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}$ are small enough.

□

As a consequence, we have the following result

Theorem 3.1 *Let* $(\boldsymbol{u}, \boldsymbol{b}, p) \in \boldsymbol{H}^{k+1}(\Omega) \times \boldsymbol{H}^{k+1}(\Omega) \times H^k(\Omega)$ *be a ssolution of the continuous MHD problem* (2.1). *Let* $(\boldsymbol{u}_h, \boldsymbol{b}_h, p_h)$ *be the solution in* $\boldsymbol{X}_h \times Q_h \times \boldsymbol{C}_h$ *of the DG scheme* (2.3). *In addition to the assumptions in Theorem 2.6, we assume that* $\frac{1}{\min(\nu,\mu)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ *and* $\frac{\kappa^{1/2}}{\nu^{1/2}\mu^{1/2}} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}$ *are small enough. Then, we have the following error estimate :*

$$\begin{aligned}
&\nu \|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{v,h} + \kappa\mu \|\boldsymbol{b}-\boldsymbol{b}_h\|^2_{m,h} \\
&\leqslant Ch^{2k}\Big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} + \|p\|^2_{H^k(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} \big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} \big)\Big)
\end{aligned} \tag{3.28}$$

with C *depending on the datas of the problem but not of the mesh size* h.

Proof. *Since* $\mathbf{u}-\boldsymbol{u}_h = \boldsymbol{\eta}_{\mathbf{u}} + \boldsymbol{\chi}_u$ *and* $\mathbf{b}-\boldsymbol{b}_h = \boldsymbol{\eta}_{\mathbf{b}} + \boldsymbol{\chi}_b$, *then the result is a direct consequence of Lemma 3.1,* (3.3) *and the triangle inequality.* □

Références

[1] H. Alfvén. *Existence of Electromagnetic-Hydrodynamic Waves*, volume 150. 1942.

[2] A. Alonso and A. Valli. A domain decomposition approach for heterogeneous time-harmonic Maxwell equations. *Comput. Methods Appl. Mech. Engrg.*, 143(1–2), 1997.

[3] C. Amrouche and V. Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czechoslovak Mathematical Journal*, 44 :109–140, 1994.

[4] C. Amrouche and N. Seloula. On the Stokes equations with the Navier-type boundary conditions. *Differ. Equ. Appl.*, 3(4) :581–607, 2011.

[5] C. Amrouche and N. El Houda Seloula. Lp-theory for vector potentials and Sobolev's inequalities for vector fields. application to the Stokes equations with pressure boundary conditions. *Mathematical Models and Methods in Applied Sciences*, 23(1) :37–92, 2013.

[6] F. Armero and J. Simo. Long-term dissipativity of time-stepping algorithms for an abstract evolution equation with applications to the incompressible MHD and Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 131 :41–90, 1996.

[7] F. Bassi, S. Rebay, G. Mariotti, S. Pedinotti, and M. Savini. A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows. *In R. Decuypere and G. Dibelius, editors, Proceedings of the 2 nd European Conference on Turbomachinery Fluid Dynamics and Thermodynamics*, pages 99–109, 1997.

[8] P. Ciarlet. *Linear and Nonlinear Functional Analysis with Applications.* Philadelphia, PA : Society for Industrial and Applied Mathematics, 2013.

[9] P.G. Ciarlet. *The finite element method for elliptic problems*, volume 4. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.

[10] B. Cockburn, G. Kanschat, D. Schötzau, and C. Schwab. Local discontinuous Galerkin methods for the Stokes system. *SIAM. J. Numer. Anal*, 40 :319–343, 2002.

[11] B. Cockburn, G.E. Karniadakis, and C.-W Shu. *Discontinuous Galerkin Methods. Theory, Computation and Applications*, volume 11. Lect. Notes Comput. Sci. Eng., Vol. pringer-Verlag, 2000.

[12] M. Costabel and M. Dauge. Weighted regularization of Maxwell equations in polyhedral domains. *Numerische Mathematik*, 93 :239–277, 2002.

[13] E. Creusé and S. Nicaise. Discrete compactness for a discontinuous Galerkin approximation of Maxwell's system. *M2AN Math. Model. Numer. Anal.*, 40(2) :413–430, 2006.

[14] P. Davidson. *An Introduction to Magnetohydrodynamics.* Cambridge University Press, 2001.

[15] D.A. Di Pietro and A. Ern. Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations. *Math. Comp.*, 79 :1303–1330, 2010.

[16] J.-F. Gerbeau. A stabilized finite element method for the incompressible magnetohydrodynamic equations. *Numer. Math.*, 87 :83–111, 2000.

[17] J.-F. Gerbeau, C.L. Bris, and T. Lelièvre. *Mathematical Methods for the Magnetohydrodynamics of Liquid Metals.* Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2006.

[18] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes equations.* Springer-Verlag, New York-Heidelberg, 1986.

[19] V. Girault, B. Rivière, and M. F. Wheeler. A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems. *Math. Comp.*, 74(249), 2005.

[20] M.D. Gunzburger, A.J. Meir, and J.S. Peterson. On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary incompressible magnetohydrodynamics. *Math. Comp*, 56(194) :523–563, 1991.

[21] U. Hasler, A. Schneebeli, and D. Schötzau. Mixed finite element approximation of incompressible MHD problems based on weighted regularization. *Appl. Numer. Math.*, 51 :19—-45, 2004.

[22] P. Houston, I. Perugia, A. Schneebeli, and D. Schötzau. Mixed discontinuous Galerkin approximation of the Maxwezll operator : The idefinite case. *ESAIM : M2AN*, 39(4) :727–753, 2005.

[23] P. Houston, I. Perugia, and D. Schötzau. *hp-DGFEM for Maxwell's equations. In : Brezzi F., Buffa A., Corsaro S., Murli A. editors.* Numerical Mathematics and Advanced Applications ENUMATH 2001. Springer-Verlage, 2003.

[24] P. Houston, I. Perugia, and D. Schötzau. Mixed discontinuous Galerkin approximation of the Maxwell operator. *SIAM Journal on Numerical Analysis*, 42(1) :434–459, 2004.

[25] P. Houston, D. Schötzau, and X. Wei. A mixed DG method for linearized incompressible magnetohydrodynamics. *J Sci Comput*, 40 :281–314, 2009.

[26] W.F. Hughes and F.J. Young. *The electromagnetics of fluids.* PWiley, New York, 1966.

[27] R. Leis. Zur theorie elektromagnetisher schwingungen in anisotropen inhomogenen medien. *Math.Z.*, 106 :213–224, 1968.

[28] S. Lohrengel and S. Nicaise. A discontinuous Galerkin method on refined meshes for the two-dimensional time-harmonic Maxwell equations in composite materials. *Journal of Computational and Applied Mathematics*, 26 :27–54, 2007.

[29] R. Moreau. *Magneto-hydrodynamics.* Kluwer Academic Publishers, 1990.

[30] I. Perugia and D. Schötzau. The hp-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations. *Math. Comp.*, 72(243) :1179–1214, 2003.

[31] J. Poirier. *Étude mathématique et approximation numérique des équations de la magnethydrdynamique avec plusieurs types de conditions aux limites.* PhD thesis, Université de Caen Normandie, 2021.

[32] W. Qiu and K. Shi. A mixed DG method and an hdg method for incompressible magnetohydrodynamics. *IMA Journal of Numerical Analysis*, 40(2) :1356–1389, 2020.

[33] W.H. Reed and T.R. Hill. Triangular mesh methods for the neutron transport equation. *Los Alamos Scientific Laboratory*, pages Technical Report (LA–UR–73–479), 1973.

[34] A. Schneebeli and D. Schötzau. Mixed finite elements for incompressible magneto-hydrodynamics. *C. R. Acad. Sci. Paris,Sér. I*, 337 :71—-74, 2003.

[35] D. Schotzau. Mixed finite element methods for stationary incompressible magneto-hydrodynamics. *Numerische Mathematik*, 96 :771–800, 2004.

[36] N. Seloula. *Mathematical Analysis and Numerical Approximation of the Stokes and Navier-Stokes Equations with Non Standard Boundary Conditions.* PhD thesis, Université de Pau et des Pays de l'Adour, 2010.

[37] J. Serrin. Mathematical principles of classical fluid mechanics. *Handbuch der physik*, 8 (part 1) :125–263, 1959.

[38] G. Stokes. *On the theories of internal friction of fluids in motion and of the equilibrium and motion of elastic solids.* PhD thesis, Trans. Camb. Phil. Soc.

[39] A.V. Usmanov. *A global numerical 3-D MHD model of the solar wind*, volume 146. 1993.

[40] P. Werner. On the exterior boundary value problem of perfect reection for stationary electromagnetic wave fields. *J.Math.Anal.Appl.*, 7 :348–396, 1963.

[41] J. Wesson. *Tokamaks.* Oxford University Press, Oxford, 2011.

[42] A. Zaghdani and C. Daveau. Two new discrete inequalities of Poincaré-Friedrichs on discontinuous spaces for Maxwell's equations. *C. R. Math. Acad. Sci. Paris*, 342(1) :29–32, 2006.