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A direct discontinuous Galerkin method for a high order nonlocal conservation law

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ABSTRACT

In this paper, we develop a Direct Discontinuous Galerkin (DDG) method for solving a time dependent partial differential equation with convection-diffusion terms and a nonlocal term which is a pseudo-differential operator of order $\alpha \in (1,2)$. This kind of equation was first introduced to describe morphodynamics of dunes and was then used for signal processing methods. We consider the DDG method which is based on the direct weak formulation of the PDE into the DG function space for both numerical solution and test functions. Suitable numerical fluxes for all operators are then introduced. We prove nonlinear stability estimates along with convergence results. Finally numerical experiments are given to illustrate qualitative behaviors of solutions and to confirm convergence results.

1. Introduction

We consider in this paper a nonlocal conservation law which appears in the formation and dynamics of sand structures such as dunes and ripples [23,25]:

$$\begin{cases} U_t + \big(f(U) - U_x + \mathcal{J}[U]\big)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ U(0,x) = U_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1}$$

where the unknown U represents the dune height and depends on the space variable x and the time variable t. $U_0 : \mathbb{R} \to \mathbb{R}$ is the initial datum, $f(U) = \frac{1}{2}U^2$ and $\mathcal{J}$ is a nonlocal operator defined by: for any Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ and any $x \in \mathbb{R}$,

$$\mathcal{J}[\varphi](x) := \int_{-\infty}^{x} |x-\xi|^{-\frac{1}{3}} \varphi'(\xi)\, d\xi. \tag{2}$$

It has been proved in [3] that the operator $\partial_x \mathcal{J}$ is a pseudo-differential operator of order $4/3$ since

$$\mathcal{F}(\partial_x \mathcal{J}[\varphi])(\xi) = -4\pi^2 \Gamma(\frac{2}{3}) \left(\frac{1}{2} - i\mathrm{sgn}(\xi)\frac{\sqrt{3}}{2} \right) |\xi|^{4/3} \mathcal{F}(\varphi)(\xi),$$

where Γ is the gamma function and $\mathcal{F}$ denotes the Fourier transform.
It will be clear from the analysis below that our results can easily be extended to the case where this nonlocal operator is replaced with a Fourier multiplier homogeneous of degree $\alpha \in (1,2)$, as in [4], and not only $\alpha = 4/3$.

Because of the opposite sign, this nonlocal operator has a deregularizing effect and thus acts as an anti-diffusive operator on the initial data. However, these instabilities are controlled by the diffusion term $-U_{xx}$ which guarantees that the initial problem (1) admits the existence and the uniqueness of a smooth solution [3]. We then always assume that there exists a sufficiently regular solution $U(t,x)$.

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Conservation laws with nonlocal or fractional terms arise in a variety of problems in finance, physical, mechanics, crow dynamics, traffic flow model etc. [32,7,5,29]. Therefore, numerical studies of this kind of equations have attracted a lot of interest in recent years. Several authors have proposed a variety of numerical schemes for solving space-fractional convection-diffusion equations. For example, a general class of difference methods for fractional conservation laws has been introduced in [22]. Finite element methods have been proposed to solve space-fractional advection equations and space and time fractional Fokker-Planck equation in [36] and [19] respectively. Finite differences approximations have been introduced for fractional advection dispersion flow equation [30] and a finite difference-quadrature approach to solve the fractional Laplacian [24]. Recently Chalon, Goatin and Villada designed a discontinuous Galerkin and Finite Volume-WENO schemes to obtain high-order approximations for nonlocal conservation laws [11].

The Discontinuous Galerkin method (DG hereafter) is a finite element method which uses a completely discontinuous piecewise polynomial space for the numerical solution and the test functions. There have been various DG methods suggested in the literature to solve equations contain higher order spatial derivatives, including the local discontinuous Galerkin (LDG) methods introduced by Cockburn and Shu in [17] and the Direct Discontinuous Galerkin (DDG hereafter) method introduced in [28] by Liu and Yan for diffusion problems. The idea of LDG methods is to rewrite the equation into a first order system and then apply the DG method to the system whereas the DDG method is based on the direct weak formulations for solutions and on appropriate numerical fluxes. In contrast to the LDG method the direct approach does not need to introduce any auxiliary variables and thus present the advantage of easier formulation and implementation and efficient computation of numerical solution.
The first DG application to fractal conservation law was studied by Cifani, Jakobsen and Karlsen in [14,15]. Xu and Hesthaven [33] applied a local discontinuous Galerkin method to fractional convection diffusion equations with a fractional Laplacian of order $\alpha \in (1,2)$ in [33]. Mustapha and McLean [31] studied a discontinuous Galerkin method for fractional diffusion and wave equations and, Deng and Hesthaven [20] a local discontinuous Galerkin method for fractional diffusion equations. Aboelenen and H. El-Hawary [2] proposed a high-order nodal discontinuous Galerkin method for a linearized fractional Cahn–Hilliard equation. Aboelenen [1] investigated a DDG method to solve equations with fractional laplacian of order $\alpha \in (1,2)$ where the equations have been expressed as a system of parabolic equation and low order integral equation.

For equations like (1) involving diffusion and fractional anti-diffusion operators, few numerical methods have been developed up to now: finite difference method [3], split-step Fourier method [9] and a finite element method [8] have been used to perform numerical simulations for the dune morphodynamics equation (1). More recently [26] proposed finite difference schemes for fractional water waves models and [10] investigated LDG schemes where they proved nonlinear stability and give errors estimates.

In this paper we propose to use a DDG method to approximate a nonlocal equation which contains a pseudo-differential operator of order 4/3 which in particularly requires to define a numerical flux for the nonlocal operator. To our knowledge, there is no work in literature which use a DDG method to numerical solve fractional equations with $\alpha \in (1,2)$. Generally a LDG approach is preferred for fractional equations of order $\alpha \in (1,2)$ which does not require to define a numerical flux for fractional operators. We choose here to consider a direct approach because it present the advantage of easier formulation and implementation and efficient computation of numerical solution.
Building on recent work of Chalon, Goatin and Villada [11] on DG methods for nonlocal conservation laws, we define a numerical flux for the nonlocal operator. Combining this with the classical approaches of the DDG methods for convection-diffusion equations, we propose a new numerical scheme for a high order nonlocal conservation law involving a pseudo-differential operators of order 4/3.

For that, we rewrite the nonlocal operator $\mathcal{J}$ to apply the DDG method and we consider suitable numerical fluxes on the cell interfaces for the convection, diffusion and nonlocal operators. We prove nonlinear stability estimates along with convergence results and we show some numerical experiments.

The rest of this paper is organized as follows. In the next section, we give some properties related to the nonlocal operator and we prove some useful lemmas. In Section 3 we introduce the semi-discret DDG method for equation (1) and we prove stability and convergence of the numerical scheme in Section 4 and Section 5 respectively. Finally, in Section 6 numerical experiments are given to illustrate qualitative behaviors of solutions and to confirm convergence results.

Notations.

- We denote by $\mathcal{F}$ the Fourier transform of f which is defined by: for all $\xi \in \mathbb{R}$,

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} e^{-2i\pi x\xi} f(x)dx.$$

  We denote by $\mathcal{F}^{-1}$ its inverse.
- We denote by $C(c_1, c_2, \cdots)$ a generic constant, strictly positive, which depends on parameters $c_1, c_2, \cdots$. The constant C is assumed to be a monotone increasing function of its arguments.

2. Preliminaries

In this section, we make some preparation including another representation of the nonlocal operator for the subsequent numerical scheme and theoretical analysis.

Lemma 2.1. *For all $\varphi \in \mathcal{S}(\mathbf{R})$ and all $x \in \mathbb{R}$,*

$$\mathcal{J}[\varphi](x) = \frac{1}{3}\int_{-\infty}^{0} \frac{\varphi(x+z)-\varphi(x)}{|z|^{4/3}}\,dz. \tag{3}$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R})$, $x, z \in \mathbb{R}$. Since

$$\varphi(x+z) - \varphi(x) = \int_{x}^{x+z} \varphi'(y)dy = \int_{0}^{1} z\,\varphi'(x+t\,z)\,dt$$

then we have

$$\int_{-\infty}^{0} \frac{\varphi(x+z)-\varphi(x)}{|z|^{4/3}}\,dz = \int_{-\infty}^{0} |z|^{-4/3}\left(\int_{0}^{1} z\,\varphi'(x+t\,z)\,dt\right)dz$$
$$= \int_{0}^{1}\left(\int_{-\infty}^{0} z|z|^{-4/3}\varphi'(x+t\,z)\,dz\right)dt$$
$$= -\int_{0}^{1}\left(\int_{-\infty}^{0} |z|^{-1/3}\varphi'(x+t\,z)\,dz\right)dt$$

and thanks to the change of variable $tz=\xi$ we get

$$\int_{-\infty}^{0} \frac{\varphi(x+z)-\varphi(x)}{|z|^{4/3}}\,dz = -\int_{0}^{1} t^{-2/3}\left(\int_{-\infty}^{0} |\xi|^{-1/3}\,\varphi'(x-\xi)\,d\xi\right)dt$$
$$= -3\int_{-\infty}^{0} |\xi|^{-1/3}\,\varphi'(x-\xi)\,d\xi$$

Therefore

$$\int_{-\infty}^{0} |\xi|^{-1/3}\,\varphi'(x-\xi)\,d\xi = -\frac{1}{3}\int_{-\infty}^{0} \frac{\varphi(x+z)-\varphi(x)}{|z|^{4/3}}\,dz.$$

Observe that the term in the left hand side is $\mathcal{J}[\varphi]$ after a simple change of variables. This completes the proof. □

From now we consider the previous representation for the operator $\mathcal{J}$ given in (3).

Lemma 2.2. *For all $\varphi \in H^{1/3}(\mathbb{R})$*

$$||\mathcal{J}[\varphi]||_{L^2(\mathbb{R})} \le \Gamma\left(\frac{2}{3}\right)||\varphi||_{H^{1/3}(\mathbb{R})}. \tag{4}$$

Proof. The proof is based on Fourier analysis arguments.
It has been proved in [3] that

$$\mathcal{F}(\mathcal{J}[\varphi])(\xi) = \Gamma\left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\,sgn(\xi)+\frac{i}{2}\right)\xi^{1/3}\mathcal{F}(\varphi)(\xi)$$

then using Parseval's inequality we have

$$||\mathcal{J}[\varphi]||^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \Gamma\left(\frac{2}{3}\right)^2 |\frac{\sqrt{3}}{2}\,sgn(\xi)+\frac{i}{2}|^2\,|\xi|^{2/3}\,|\mathcal{F}(\varphi)(\xi)|^2\,d\xi$$
$$= \Gamma\left(\frac{2}{3}\right)^2 \int_{\mathbb{R}} |\xi|^{2/3}\,|\mathcal{F}(\varphi)(\xi)|^2\,d\xi$$
$$\le \Gamma\left(\frac{2}{3}\right)^2 \int_{\mathbb{R}} (1+|\xi|^2)^{1/3}\,|\mathcal{F}(\varphi)(\xi)|^2\,d\xi$$
$$= \Gamma\left(\frac{2}{3}\right)^2 ||\varphi||^2_{H^{1/3}(\mathbb{R})}. \quad \square$$

Let us now introduce a partition of the domain consisting of cells $I_j=(x_{j-1/2},x_{j+1/2})$ for all $j\in\mathbb{Z}$. We denote the cell lengths $\Delta x_j = x_{j+1/2}-x_{j-1/2}$ and we define $\Delta x = \max_j \Delta x_j$.
We denote by $P^k(I_j)$ the space of all polynomials of degree at most k with support on I_j, and we define the piecewise polynomial space V^k as

$$V^k = \left\{ v; v_{|I_j} P^k(I_j) \text{ for all } j\in\mathbb{Z}\right\}.$$

Let us introduce the operators

$$[v(x_{j+1/2})] = v(x^+_{j+1/2}) - v(x^-_{j+1/2}), \qquad \overline{v(x_{j+1/2})} = \frac{1}{2}(v(x^+_{j+1/2}) + v(x^-_{j+1/2})).$$

The approximate solutions are sought under the form

$$u(t,x)_{|I_j} = \sum_{l=0}^{k} c^l_j(t)\phi^l_j(x)$$

where c_j^l are the degrees of freedom in the element I_j (the unknown) and $\left\{\phi_j^l\right\}_{l=0,\cdots,k}$ constitutes a basis of $P^k(I_j)$.

Lemma 2.3. *Let $\varphi \in V^k \cap L^2(\mathbb{R})$ and $0 < r < 1$. Then $\varphi \in H^{1/3}(\mathbb{R})$ and*

$$|\varphi|^2_{H^{1/3}(\mathbb{R})} \le C(r^{-2/3})||\varphi||^2_{L^2(\mathbb{R})} + C(r^{1/3})\sum_{j\in\mathbb{Z}}[\varphi]_j^2 + C(r^{4/3})\sum_{j\in\mathbb{Z}}||\varphi'||^2_{L^2(I_j)},$$

where $|u|_{H^{\lambda/2}} := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(z)-u(x))^2}{|z-x|^{1+\lambda}}\,dz\,dx$ is the semi-norm of the fractional Sobolev space $H^{\lambda/2}$.

Proof. Let us consider a function $\varphi \in V^k \cap L^2(\mathbb{R})$. Then it has been proved in [15, Lemma A.4] that for $|z| < 1$

$$\int_{\mathbb{R}} (\varphi(x+z)-\varphi(x))^2\,dx \le C\left(|z|\sum_{j\in\mathbb{Z}}[\varphi]_j^2 + |z|^2\sum_{j\in\mathbb{Z}}||\varphi'||^2_{L^2(I_j)}\right). \tag{5}$$

Therefore for $r \in (0,1)$ we have

$$\begin{aligned}|\varphi|^2_{H^{1/3}(\mathbb{R})} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\varphi(x+z)-\varphi(x))^2}{|z|^{5/3}}\,dx\,dz\\ &= \int_{|z|<r}\int_{\mathbb{R}} \frac{(\varphi(x+z)-\varphi(x))^2}{|z|^{5/3}}\,dx\,dz + \int_{|z|>r}\int_{\mathbb{R}} \frac{(\varphi(x+z)-\varphi(x))^2}{|z|^{5/3}}\,dx\,dz\\ &= T_1 + T_2\end{aligned}$$

For the first term T_1 we use the estimate (5) and we obtain

$$\begin{aligned}T_1 &\le C\left(\sum_{j\in\mathbb{Z}}[\varphi]_j^2 \int_{|z|<r} \frac{dz}{|z|^{2/3}}\,dz + \sum_{j\in\mathbb{Z}}||\varphi'||^2_{L^2(I_j)} \int_{|z|<r} |z|^{1/3}\,dz\right)\\ &\le C(r^{1/3})\sum_{j\in\mathbb{Z}}[\varphi]_j^2 + C(r^{4/3})\sum_{j\in\mathbb{Z}}||\varphi'||^2_{L^2(I_j)}.\end{aligned}$$

For the term T_2 we directly get

$$\begin{aligned}T_2 &\le 4||\varphi||^2_{L^2(\mathbb{R})}\int_{|z|>r}\frac{dz}{|z|^{5/3}}\,dz\\ &\le C(r^{-2/3})||\varphi||^2_{L^2(\mathbb{R})}.\end{aligned}$$

Combining the previous estimates, we obtain (5). □

The following lemma of Gronwall type will be used:

Lemma 2.4. *([6, Lemma 3.3], [21, Chapter 2]). Let $y, q, z, r \in C([0,T])$ be nonnegative functions and let*

$$y(t) + q(t) \le z(t) + \int_0^t r(s)y(s)\,ds, \quad \forall t \in [0,T].$$

Then

$$y(t) + q(t) \le z(t) + \int_0^t r(\theta)z(\theta)\exp\left(\int_\theta^t r(s)\,ds\right)d\theta, \quad \forall t \in [0,T].$$

We will also need to use the following inverse property [13]: For any function $w_h \in V^k$, the following inequalities hold

$$\begin{aligned}&||w_h||_\infty \le C\Delta x^{-1/2}||w_h||_{L^2}\\ &||w_h||_{\Gamma_h} \le C\Delta x^{-1/2}||w_h||_{L^2}\end{aligned} \tag{6}$$

where Γ_h denotes the set of interface points of all the elements.

3. Formulation of the scheme

In this section we introduce the numerical scheme for the dune morphodynamics model (1). We first multiply the equation by an arbitrary $v \in P^k(I_j)$, integrate over I_j, and have integration by parts. Then, we define the discrete DDG scheme as follows: Seek $u \in V^k \cap L^2(\mathbb{R})$ such that

$$\int_{I_j} u_t v\,dx - \int_{I_j} f(u)v_x\,dx + \hat{f}(u_{j+1/2})v^-_{j+\frac{1}{2}} - \hat{f}(u_{j+\frac{1}{2}})v^+_{j-\frac{1}{2}} + \int_{I_j} u_x v_x\,dx$$

$$-\hat{u}_x(x_{j+\frac{1}{2}})v^-_{j+\frac{1}{2}} + \hat{u}_x(x_{j-\frac{1}{2}})v^+_{j-\frac{1}{2}} - \int_{I_j} \mathcal{J}[u]v_x\,dx + \widehat{\mathcal{J}[u]}(x_{j+\frac{1}{2}})v^-_{j+\frac{1}{2}} - \widehat{\mathcal{J}[u]}(x_{j-\frac{1}{2}})v^+_{j-\frac{1}{2}} = 0$$

$$\int_{I_j} u(0,x)v(x)\,dx = \int_{I_j} U_0(x)v(x)\,dx, \tag{7}$$

where the numerical fluxes are given by

- *Convection term:* We consider the Lipschitz continuous E-flux (a consistent and monotone flux)

  $$\hat{f}(u_{j+1/2}) = \hat{f}(u(x^-_{j+1/2}), u(x^+_{j+1/2})).$$

  Note that since $\hat{f}$ is consistent i.e. $\hat{f}(u,u) = f(u)$ and monotone i.e. increasing w.r.t. its first variable and decreasing w.r.t. its second variable,

  $$\int_{u^-_{j+1/2}}^{u^+_{j+1/2}} \left(f(x) - \hat{f}(u^-_{j+1/2}, u^+_{j+1/2})\right) dx \geq 0.$$

- *Diffusion term:* We follow Liu and Yan [28] and we introduce the flux

  $$\hat{u}_x = \beta_0 \frac{[u]}{\Delta x} + \overline{u}_x \tag{8}$$

  which satisfies the following admissibility condition

  Définition 3.1 *(Admissibility [28]).* The numerical flux $\hat{u_x}$ is admissible if there exist a $\gamma \in (0,1)$ and $\alpha > 0$ such that

  $$\gamma \sum_j \int_{I_j} u_x^2\,dx + \sum_j (\hat{u_x})_{j+\frac{1}{2}}[u]_{j+\frac{1}{2}} \geq \alpha \sum_j \frac{[u]^2_{j+\frac{1}{2}}}{\Delta x} \tag{9}$$

  holds for any piecewise polynomials of degree k.

- *Nonlocal term:* For the nonlocal term, we use the expression (3) and we define

  $$\widehat{\mathcal{J}[u]}_{j+\frac{1}{2}} = \sum_{i\leq j} \int_{I_i} \frac{u(z) - u(x_{j+1/2})}{|z - x_{j+1/2}|^{4/3}}\,dz. \tag{10}$$

  Similar numerical fluxes have been considered for nonlocal scalar conservation laws in [11].

4. The nonlinear stability

In this section, we discuss stability of the proposed scheme.
Let us first review the stability property for the continuous problem. Using Fourier analysis it has been proved in [3] that the exact solution of the initial value problem (1) satisfies

$$||U(t,\cdot)||_{L^2(\mathbb{R})} \leq e^{w_* t}||U_0||_{L^2(\mathbb{R})}, \qquad \forall t \in (0,T) \tag{11}$$

where w_* is a positive constant.
Therefore, we say that the DDG scheme (7) is L^2-stable if the numerical solution u satisfies

$$||u(T,\cdot)||_{L^2(\mathbb{R})} \leq C(T)||u_0||_{L^2(\mathbb{R})}.$$

In the following Theorem, we show that the proposed DDG scheme (7) is stable.

Theorem 4.1 *(Energy stability). Consider the DDG scheme (7) with the numerical fluxes defined in Section 3. Then we have for small Δx,*

$$||u(T,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma)\int_0^T \sum_{j\in\mathbb{Z}} \int_{I_j} u_x^2(s,x)\,dx\,ds + \alpha \int_0^T \sum_{j\in\mathbb{Z}} \frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}\,ds \leq e^{C(\alpha,\gamma)T}||u_0||^2_{L^2(\mathbb{R})}. \tag{12}$$

Proof. Let us first some over all j and set $v = u$ in the numerical scheme (7):

$$\int_{\mathbb{R}} u_t u + \sum_{j\in\mathbb{Z}} \left([\Phi(u)]_{j+1/2} - (\hat{f}[u])_{j+\frac{1}{2}}\right) + \sum_{j\in\mathbb{Z}} \int_{I_j} u_x^2 + \sum_{j\in\mathbb{Z}} \left(\hat{u}_x[u]\right)_{j+\frac{1}{2}} - \sum_{j\in\mathbb{Z}} \left(\widehat{\mathcal{J}[u]}[u]\right)_{j+\frac{1}{2}} - \sum_{j\in\mathbb{Z}} \int_{I_j} \mathcal{J}[u]u_x = 0$$

where $\Phi(u) = \int^u f(u)\,du$.
Thanks to the monotone property of flux $\hat{f}$ we have $[\Phi(u)]_{j+1/2} - (\hat{f}[u])_{j+\frac{1}{2}} > 0$.
Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2_{L^2(\mathbb{R})} + \sum_{j\in\mathbb{Z}}\int_{I_j} u_x^2 + \sum_{j\in\mathbb{Z}}\left(\widehat{u}_x[u]\right)_{j+\frac{1}{2}} - \sum_{j\in\mathbb{Z}}\left(\widehat{\mathcal{J}[u]}[u]\right)_{j+\frac{1}{2}} - \sum_{j\in\mathbb{Z}}\int_{I_j}\mathcal{J}[u]u_x \le 0.$$

Now using admissibility condition (9) there exist a $\gamma\in(0,1)$ and $\alpha>0$ such that

$$\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma)\sum_{j\in\mathbb{Z}}\int_{I_j} u_x^2(t,x)dx + \alpha\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}$$

$$\le \sum_{j\in\mathbb{Z}}\int_{I_j}\mathcal{J}[u]u_x + \sum_{j\in\mathbb{Z}}\left(\widehat{\mathcal{J}[u]}[u]\right)_{j+\frac{1}{2}}$$

$$\le \frac{4}{1-\gamma}\sum_{j\in\mathbb{Z}}||\mathcal{J}[u]||^2_{L^2(I_j)} + \frac{1-\gamma}{4}\sum_{j\in\mathbb{Z}}||u_x||^2_{L^2(I_j)} + \frac{4\Delta x}{\alpha}\sum_{j\in\mathbb{Z}}\widehat{\mathcal{J}[u]}^2_{j+\frac{1}{2}} + \frac{\alpha}{4}\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}$$

Using Lemma 2.2, Lemma 2.3 and inverse inequality we obtain

$$\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma)\sum_{j\in\mathbb{Z}}\int_{I_j} u_x^2(t,x)dx + \alpha\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}$$

$$\le \left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)||\mathcal{J}[u]||^2_{L^2(\mathbb{R})} + \frac{1-\gamma}{4}\sum_{j\in\mathbb{Z}}||u_x||^2_{L^2(I_j)} + \frac{\alpha}{4}\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}$$

$$\le \left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)C(r^{1/3})\sum_{j\in\mathbb{Z}}[u]^2_{j+\frac{1}{2}} + \left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)C(r^{4/3})\sum_{j\in\mathbb{Z}}||u_x||^2_{L^2(I_j)}$$

$$+\left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)C(r^{-2/3})||u||^2_{L^2(\mathbb{R})} + \frac{\alpha}{4}\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x} + \frac{1-\gamma}{4}\sum_{j\in\mathbb{Z}}||u_x||^2_{L^2(I_j)}$$

which gives for small Δx

$$\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2_{L^2(\mathbb{R})} + \frac{3}{4}(1-\gamma)\sum_{j\in\mathbb{Z}}\int_{I_j} u_x^2(t,x)dx + \frac{3}{4}\alpha\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}$$

$$\le \left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)C(r^{1/3})\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x} + \left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)C(r^{4/3})\sum_{j\in\mathbb{Z}}||u_x||^2_{L^2(I_j)}$$

$$+\left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)C(r^{-2/3})||u||^2_{L^2(\mathbb{R})}.$$

For r chosen in a way that the following conditions are satisfied

$$\begin{cases} C(r^{1/3})\left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)\le\frac{\alpha}{4} \\ C(r^{4/3})\left(\frac{4}{1-\gamma}+\frac{4}{\alpha}\right)\le\frac{1-\gamma}{4}\end{cases} \tag{13}$$

we finally obtain

$$\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2_{L^2(\mathbb{R})} + \frac{1-\gamma}{2}\sum_{j\in\mathbb{Z}}\int_{I_j} u_x^2(t,x)dx + \frac{\alpha}{2}\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x} \le C(\alpha,\gamma)||u||^2_{L^2(\mathbb{R})}$$

Integrating the previous inequality from 0 to $t\in[0,T]$ yields

$$||u(t,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma)\int_0^t\sum_{j\in\mathbb{Z}}\int_{I_j}u_x^2(s,x)dx\,ds + \alpha\int_0^t\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}ds \le C(\alpha,\gamma)\int_0^t||u(s,\cdot)||^2_{L^2(\mathbb{R})}ds + ||u_0||^2_{L^2(\mathbb{R})}$$

Now we apply Gronwall Lemma 2.4 with

$$y(t)=||u(t,\cdot)||^2_{L^2(\mathbb{R})},\quad q(t)=(1-\gamma)\int_0^t\sum_{j\in\mathbb{Z}}\int_{I_j}u_x^2(s,x)dx\,ds + \alpha\int_0^t\sum_{j\in\mathbb{Z}}\frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}ds,$$

$r(t)=C(\alpha,\gamma)$ and $z(t)=||u_0||^2_{L^2(\mathbb{R})}$ and we get

$$||u(t,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma)\int_0^t \sum_{j\in\mathbb{Z}} \int_{I_j} u_x^2(s,x)dx\,ds + \alpha \int_0^t \sum_{j\in\mathbb{Z}} \frac{[u]^2_{j+\frac{1}{2}}}{\Delta x}\,ds \le ||u_0||^2_{L^2(\mathbb{R})} + \int_0^t C(\alpha,\gamma)||u_0||^2_{L^2(\mathbb{R})} \exp\Big(\int_\theta^t C(\alpha,\gamma)ds\Big)\,d\theta$$

$$\le ||u_0||^2_{L^2(\mathbb{R})} + C(\alpha,\gamma)\,||u_0||^2_{L^2(\mathbb{R})} \int_0^t e^{C(\alpha,\gamma)(t-\theta)}\,d\theta$$

$$\le e^{C(\alpha,\gamma)t}||u_0||^2_{L^2(\mathbb{R})}.$$

The proof of Theorem 4.1 is now complete. □

5. Error estimates

Inspired by the stability estimate (12), we introduce the following energy norm to measure the error

$$|||v(t,\cdot)||| := \left(||v(t,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma)\int_0^t \sum_{j\in\mathbb{Z}} \int_{I_j} v_x^2(t,x)dx\,dt + \alpha \int_0^t \sum_{j\in\mathbb{Z}} \frac{[v]^2_{j+\frac{1}{2}}}{\Delta x} \right)^{1/2}.$$

Moreover, let $\mathbb{P}$ be the L^2 projection defined as

$$\int_{\mathbb{R}} (\mathbb{P}(U)(x) - U(x))v(x)\,dx = 0, \qquad \forall v \in V^k \cap L^2(\mathbb{R}). \tag{14}$$

We then consider the following projection properties stated in [28, Lemma 3.1].

Lemma 5.1 *([28]). Let $U \in H^{s+1}(I_j)$ for $j = 0, \cdots, M-1$, and $s \ge 0$. Then we have the following estimates:*

1. $|\mathbb{P}(U) - U|_{m,I_j} \le c_k \Delta x^{\min(k,s)+1-m} |U|_{s+1,I_j}, \quad m \le k+1$
2. $|\partial_x^m(\mathbb{P}(U) - U)_{x+1/2}| \le c_k \Delta x^{\min(k,s)+1/2-m} |U|_{s+1,I_{j+1/2}}, \quad m \le k+1/2,$

where $m \ge 0$ is an integer and $|\cdot|_{m,I_j}$ denotes the semi-norm of $H^m(I_j)$.

5.1. *The linear case $f = cU$*

In this subsection, we consider the linear problem

$$\begin{cases} U_t + \big(cU - U_x + \mathcal{J}[U]\big)_x = 0, \\ U(0,x) = U_0(x). \end{cases} \tag{15}$$

For the convection term, we opt for the well-known monotone Lax-Friedrich flux [18]

$$\hat{f}(u^-_{j+1/2}, u^+_{j+1/2}) = c\overline{u_{j+1/2}} - |c|\frac{[u]_{j+1/2}}{2}. \tag{16}$$

We then define the bilinear form associated to the numerical scheme (7) by

$$\mathcal{B}(w,v) := \mathcal{B}_l(w,v) + \mathcal{B}_f(w,v)$$

where

$$\mathcal{B}_l(w,v) := \int_{\mathbb{R}} w_t(t,x)v(t,x)\,dx + \sum_{j\in\mathbb{Z}} \int_{I_j} w_x(t,x)v_x(t,x)\,dx$$
$$- \sum_{j\in\mathbb{Z}} \int_{I_j} \mathcal{J}[w]v_x(t,x)\,dx - \sum_{j\in\mathbb{Z}} (\widehat{\mathcal{J}[w]}[v])_{j+\frac{1}{2}} + \sum_{j\in\mathbb{Z}} (\hat{w}_x[v])_{j+\frac{1}{2}} \tag{17}$$

and

$$\mathcal{B}_f(w,v) = -\sum_{j\in\mathbb{Z}} \left[(\hat{f}(w)[v])_{j+\frac{1}{2}} + \int_{I_j} f(w)v_x \right].$$

Lemma 5.2. *Let U be a regular solution of (15). Then we have for all $v \in V^k \cap L^2(\mathbb{R})$*

$$\mathcal{B}(U,v) = 0.$$

Proof. Let $v \in V^k \cap L^2(\mathbb{R})$. We first multiply (15) by v, integrate over I_j and have integration by parts

$$\int_{I_j} U_t v\, dx - \int_{I_j} f(U) v_x\, dx + (f(U)v)(x_{j+\frac{1}{2}}) - (f(U)v)(x_{j-\frac{1}{2}}) + \int_{I_j} U_x v_x\, dx$$
$$- (U_x v)(x_{j+\frac{1}{2}}) + (U_x v)(x_{j-\frac{1}{2}}) - \int_{I_j} \mathcal{J}[U] v_x\, dx + (\mathcal{J}[u]\, v)(x_{j+\frac{1}{2}}) - (\mathcal{J}[u]\, v)(x_{j-\frac{1}{2}}) = 0 \tag{18}$$

Moreover, since U is regular, we have $[U]_{j+1/2} = 0, \overline{(U_x)}_{j+1/2} = U_x(x_{j+1/2})$ then

$$U_x(x_{j+1/2}) = \widehat{(U_x)}_{j+1/2}, \quad f(U)(x_{j+1/2}) = \hat{f}(U)_{j+1/2}$$

and from definition (10) of $\hat{\mathcal{J}}$, we have $\widehat{\mathcal{J}[U]}_{j+1/2} = \mathcal{J}[U](x_{j+1/2})$. We have similar results for $x_{j-1/2}$. Therefore by summing over all I_j, (18) gives $\mathcal{B}(U, v) = 0$. $\square$

Theorem 5.3 *(Convergence). Let U be a regular solution of (15) where all terms are well defined and $u \in C^1([0,T]; V^k \cap L^2(\mathbb{R}))$ be a solution of (7) with the numerical fluxes defined in section 3. With $e := u - U$, we have for small Δx the following error estimate:*

$$|||e(T,\cdot)||| \le C \Delta x^k |U(T,\cdot)|_{k+1}, \tag{19}$$

where $C = C(k, \gamma, \alpha, c, T)$ is a constant depending on k, γ, α, c, T.

Proof. Let $e = u - U = \mathbb{P}(e) - (U - \mathbb{P}(U))$. Then, we have

$$|||e(T,\cdot)||| \le |||\mathbb{P}(e)(T,\cdot)||| + |||(U - \mathbb{P}(U))(T,\cdot)|||. \tag{20}$$

From Lemma 5.1 it is sufficient to estimate the first term in the right side of (20) since

$$|||(U - \mathbb{P}(U))(T,\cdot)||| \le C |U(T,\cdot)|_{k+1} (\Delta x)^k.$$

Since for any v in $V^k \cap L^2(\mathbb{R})$, the numerical solution u satisfies $\mathcal{B}(u, v) = 0$ and the exact solution U satisfies $\mathcal{B}(U, v) = 0$ thanks to Lemma 5.2, we have

$$\mathcal{B}(u, v) - \mathcal{B}(U, v) = \mathcal{B}_l(u, v) + \mathcal{B}_f(u, v) - \mathcal{B}_l(U, v) - \mathcal{B}_f(U, v) = 0.$$

Then for $v = \mathbb{P}(e)$ and since $\mathbb{P}(e) = u - \mathbb{P}(U)$ we have

$$\mathcal{B}_l(\mathbb{P}(e), \mathbb{P}(e)) = \mathcal{B}_l(U - \mathbb{P}(U), \mathbb{P}(e)) + \mathcal{B}_f(U, \mathbb{P}(e)) - \mathcal{B}_f(u, \mathbb{P}(e)).$$

From the admissibility property (9) there exist $\gamma \in (0,1)$ and $\alpha > 0$ such that

$$\mathcal{B}_l(\mathbb{P}(e), \mathbb{P}(e)) \ge \frac{1}{2}\frac{d}{dt} ||\mathbb{P}(e)(t,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma) \sum_{j\in\mathbb{Z}} ||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \alpha \sum_{j\in\mathbb{Z}} \frac{[\mathbb{P}(e)]^2_{j+\frac{1}{2}}}{\Delta x}$$
$$- \sum_{j\in\mathbb{Z}} \int_{I_j} \mathcal{J}[\mathbb{P}(e)](\mathbb{P}(e))_x\, dx - \sum_{j\in\mathbb{Z}} (\widehat{\mathcal{J}[\mathbb{P}e]}[\mathbb{P}e])_{j+\frac{1}{2}}.$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt} ||\mathbb{P}(e)(t,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma) \sum_{j\in\mathbb{Z}} ||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \alpha \sum_{j\in\mathbb{Z}} \frac{[\mathbb{P}(e)]^2_j}{\Delta x} \le T_1 + T_2 + T_3 + T_4 \tag{21}$$

where

$$T_1 = \mathcal{B}_l(U - \mathbb{P}(U), \mathbb{P}(e))$$
$$T_2 = \mathcal{B}_f(U, \mathbb{P}(e)) - \mathcal{B}_f(u, \mathbb{P}(e))$$
$$T_3 = \sum_{j\in\mathbb{Z}} \int_{I_j} \mathcal{J}[\mathbb{P}(e)](\mathbb{P}(e))_x\, dx$$
$$T_4 = \sum_{j\in\mathbb{Z}} (\widehat{\mathcal{J}[\mathbb{P}e]}[\mathbb{P}e])_{j+\frac{1}{2}}$$

Let us first study the term T_1. Using Hölder inequality we have

$$T_1 = \mathcal{B}_l(U - \mathbb{P}(U), \mathbb{P}(e)) = \sum_{j\in\mathbb{Z}} \int_{I_j} (U - \mathbb{P}(U))_x \mathbb{P}(e)_x\, dx + \sum_{j\in\mathbb{Z}} (\widehat{U - \mathbb{P}(U)})_x [\mathbb{P}(e)]_{j+\frac{1}{2}}$$
$$- \sum_{j\in\mathbb{Z}} \int_{I_j} \mathcal{J}[U - \mathbb{P}(U)] \mathbb{P}(e)_x\, dx - \sum_{j\in\mathbb{Z}} \mathcal{J}[\widehat{U - \mathbb{P}(U)}]_{j+\frac{1}{2}} [\mathbb{P}(e)]_{j+\frac{1}{2}}$$

$$\le \frac{10}{1-\gamma}\sum_{j\in\mathbb{Z}}||(U-\mathbb{P}(U))_x||^2_{L^2(I_j)} + \frac{1-\gamma}{10}\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \frac{12\Delta x}{\alpha}\sum_{j\in\mathbb{Z}}(\widehat{U-\mathbb{P}(U)})^2_x$$

$$+\frac{\alpha}{12}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac12}}{\Delta x} + \frac{10}{1-\gamma}\sum_{j\in\mathbb{Z}}||\mathcal{J}[U-\mathbb{P}(U)]||^2_{L^2(I_j)} + \frac{1-\gamma}{10}\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)}$$

$$+\frac{12\Delta x}{\alpha}\sum_{j\in\mathbb{Z}}\mathcal{J}[\widehat{U-\mathbb{P}(U)}]^2_{j+\frac12} + \frac{\alpha}{12}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac12}}{\Delta x}$$

$$\le \frac{10}{1-\gamma}\sum_{j\in\mathbb{Z}}||(U-\mathbb{P}(U))_x||^2_{L^2(I_j)} + \frac{12\Delta x}{\alpha}\sum_{j\in\mathbb{Z}}(\widehat{U-\mathbb{P}(U)})^2_x + \frac{10}{1-\gamma}\sum_{j\in\mathbb{Z}}||\mathcal{J}[U-\mathbb{P}(U)]||^2_{L^2(I_j)}$$

$$+\frac{12\Delta x}{\alpha}\sum_{j\in\mathbb{Z}}\mathcal{J}[\widehat{U-\mathbb{P}(U)}]^2_{j+\frac12} + \frac{1-\gamma}{5}\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \frac{\alpha}{6}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac12}}{\Delta x}.$$

For the term T_2, since $e = \mathbb{P}(e) - (U - \mathbb{P}(U))$, we have,

$$T_2 = \mathcal{B}_f(-e, \mathbb{P}(e)) = \mathcal{B}_f(U-\mathbb{P}(U), \mathbb{P}(e)) - \mathcal{B}_f(\mathbb{P}(e), \mathbb{P}(e))$$

and since the monotone numerical flux (16) is a quadratic entropy flux (see [18,27]):

$$\mathcal{B}_f(\mathbb{P}(e), \mathbb{P}(e)) \ge 0 \tag{22}$$

then

$$T_2 \le \mathcal{B}_f(U-\mathbb{P}(U), \mathbb{P}(e))$$

Moreover from the definition of L^2 projection (14) and Hölder's inequality we obtain

$$|T_2| \le \sum_{j\in\mathbb{Z}}\hat{f}(U-\mathbb{P}(U))_{j+1/2}[\mathbb{P}(e)]_{j+\frac12}$$

$$\le \frac{12}{\alpha}\Delta x\sum_{j\in\mathbb{Z}}\hat{f}(U-\mathbb{P}(U))^2_{j+1/2} + \frac{12}{\alpha}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac12}}{\Delta x}$$

Using now [16, Lemma 2.16], we have

$$|T_2| \le \frac{12}{\alpha}c_k\Delta x^{2k+2}|U(t,\cdot)|^2_{k+1} + \frac{\alpha}{12}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac12}}{\Delta x}$$

where the constant c_k depends solely on k.

For the third term T_3, thanks to Lemma 2.2 and Lemma 2.3 we obtain

$$|T_3| \le \frac{5}{1-\gamma}\sum_{j\in\mathbb{Z}}||\mathcal{J}[\mathbb{P}(e)]||^2_{L^2(I_j)} + \frac{1-\gamma}{5}\sum_{j\in\mathbb{Z}}||(\mathbb{P}(e))_x||^2_{L^2(I_j)}$$

$$\le \frac{5}{1-\gamma}C(r^{-2/3})||\mathbb{P}(e)||^2_{L^2(\mathbb{R})} + \frac{5}{1-\gamma}C(r^{1/3})\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac12}}{\Delta x}$$

$$+\frac{5}{1-\gamma}C(r^{4/3})\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \frac{1-\gamma}{5}\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)}.$$

Again from Lemma 2.2, Lemma 2.3 and inverse inequality (6) we get the following estimate for T_4

$$|T_4| \le 12\frac{\Delta x}{\alpha}\sum_{j\in\mathbb{Z}}\widehat{\mathcal{J}[\mathbb{P}(e)]}^2_{j+\frac12} + \frac{\alpha}{12}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_j}{\Delta x}$$

$$\le \frac{12}{\alpha}C(r^{-2/3})||\mathbb{P}(e)||^2_{L^2(\mathbb{R})} + \frac{12}{\alpha}C(r^{1/3})\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac12}}{\Delta x} + \frac{\alpha}{12}C(r^{4/3})\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)}$$

$$+\frac{\alpha}{12}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac12}}{\Delta x}.$$

We then choose $r \in (0,1)$ such that the following condition are satisfied

$$\begin{cases} C(r^{1/3})\left(\dfrac{10}{1-\gamma}+\dfrac{12}{\alpha}\right) \le \dfrac{\alpha}{12} \\ C(r^{4/3})\left(\dfrac{10}{1-\gamma}+\dfrac{12}{\alpha}\right) \le \dfrac{1-\gamma}{10} \end{cases}$$

and substituting the above estimates for all terms $T_i, i = 1,2,3,4$ into (21), we obtain for small Δx

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\mathbb{P}(e)(t,\cdot)||^2_{L^2(\mathbb{R})} + (1-\gamma)\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \alpha\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac{1}{2}}}{\Delta x}\\
&\le \frac{10}{1-\gamma}\sum_{j\in\mathbb{Z}}||(U-\mathbb{P}(U))_x||^2_{L^2(I_j)} + \frac{12\Delta x}{\alpha}\sum_{j\in\mathbb{Z}}\widehat{(U-\mathbb{P}(U))}^2_x + \frac{10}{1-\gamma}\sum_{j\in\mathbb{Z}}||\mathcal{J}[U-\mathbb{P}(U)]||^2_{L^2(I_j)}\\
&+\frac{12\Delta x}{\alpha}\sum_{j\in\mathbb{Z}}\mathcal{J}[\widehat{U-\mathbb{P}(U)}]_{j+\frac{1}{2}} + C(\alpha)||\mathcal{J}[U(t,\cdot)||_{L^2(\mathbb{R})} + C(k,\gamma,\alpha,c)\Delta x^{2k}|U|^2_{k+1}\\
&+\frac{1-\gamma}{2}\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \frac{\alpha}{2}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac{1}{2}}}{\Delta x}.
\end{aligned}
$$

Finally from Lemma 2.2, Lemma 2.3, Lemma 5.1 and Gronwall's Lemma, we obtain

$$|||\mathbb{P}(e)(T,\cdot)|||^2 \le C(k,\gamma,\alpha,c,T)(\Delta x)^{2k}|U(T,\cdot)|^2_{k+1}. \quad \square$$

5.2. Nonlinear case

Let us now assume that f is nonlinear and let us prove the convergence of the proposed scheme (7).

To deal with the nonlinear term f, we argue as [12,34] by defining

$$\sum_{j\in\mathbb{Z}}\mathcal{H}_j(f;U,u,v) = \sum_{j\in\mathbb{Z}}\int_{I_j}(f(U)-f(u))v_x\,dx + \sum_{j\in\mathbb{Z}}(f(U)-\hat{f})[v]_{j+\frac{1}{2}}$$

and we use the following result:

Lemma 5.4 *([34]). For $\mathcal{H}_j(f;U,u,v)$ defined above, we have the following estimate:*

$$
\begin{aligned}
\sum_j \mathcal{H}_j(f;U,u,v) &\le -\frac{1}{4}\kappa(\hat{f};u)[v]^2 + (C + C_\star(||v||_\infty + \Delta x^{-1}||e_u||^2_\infty))||v||^2_{L^2}\\
&+ (C + C_\star\Delta x^{-1}||e_u||^2_\infty)\Delta x^{2k},
\end{aligned}
$$

with $e_u = U - u$ and

$$\kappa(\hat{f};w) \equiv \kappa(\hat{f};w^-,w^+) := \begin{cases} [w]^{-1}(f(\overline{w}) - \hat{f}(w)) & \text{if } [w]\ne 0\\ \frac{1}{2}|f'(\overline{w})| & \text{if } [w]=0.\end{cases}$$

It has been proved that $\kappa(\hat{f};w)$ is nonnegative and bounded for any $(w^-,w^+)\in\mathbb{R}^2$ [35].

Following the lines of [12,34], we assume that for Δx small and $k\ge 1$ the following assumption

$$||U-u||_{L^2(\mathbb{R})} \le \Delta x. \tag{23}$$

$$||e_u||_\infty \le C\Delta x^{1/2} \text{ and } ||\mathbb{P}U - u||_\infty \le C\Delta x^{1/2} \tag{24}$$

Theorem 5.5. *Let U be a regular solution of (1) and $u \in C^1([0,T]; V^k \cap L^2(\mathbb{R}))$ be the discrete solution of the DDG scheme (7) with the numerical fluxes defined in section 3. We have for Δx small enough satisfying (23)-(24) and $k\ge 1$,*

$$|||U-u||| \le C\Delta x^k.$$

Proof. The proof follows the same lines as the previous proof for the linear case. Only the term T_2 needs to be treated differently. Indeed, using previous notations T_2 can be rewritten as:

$$T_2 = \mathcal{B}_f(U,\mathbb{P}(e)) - \mathcal{B}_f(u,\mathbb{P}(e)) = \sum_{j\in\mathbb{Z}}\mathcal{H}_j(f;U,u,\mathbb{P}(e))$$

As previously, for $r\in(0,1)$ judiciously chosen, we obtain for small Δx

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\mathbb{P}(e)(t,\cdot)||^2_{L^2(\mathbb{R})} + \frac{1-\gamma}{2}\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \frac{\alpha}{2}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac{1}{2}}}{\Delta x}\\
&\le C(k,\gamma,\alpha)\Delta x^{2k}|U|^2_{k+1} + \sum_{j\in\mathbb{Z}}\mathcal{H}_j(f;U,u,\mathbb{P}(e)).
\end{aligned}
$$

Using Lemma 5.4, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\mathbb{P}(e)(t,\cdot)||^2_{L^2(\mathbb{R})} + \frac{1-\gamma}{2}\sum_{j\in\mathbb{Z}}||\mathbb{P}(e)_x||^2_{L^2(I_j)} + \frac{\alpha}{2}\sum_{j\in\mathbb{Z}}\frac{[\mathbb{P}(e)]^2_{j+\frac{1}{2}}}{\Delta x}\\
&\le C(k,\gamma,\alpha)\Delta x^{2k}|U|^2_{k+1} + (C + C_\star(||\mathbb{P}(e)||_\infty + \Delta x^{-1}||e_u||^2_\infty))||\mathbb{P}(e)||^2_{L^2} + (C + C_\star\Delta x^{-1}||e_u||^2_\infty)\Delta x^{2k}.
\end{aligned}
$$

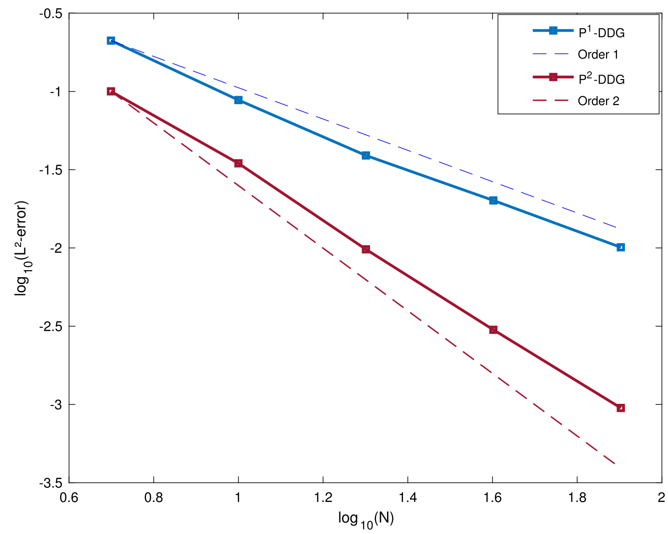

Fig. 1. Test 1: Convergence curves for $k=1$ (solid blue line) and $k=2$ (solid red line). Dashed lines represent slopes of order 1 (in blue) and of order 2 (in red).

We then estimate $|||\mathbb{P}(e)(T,\cdot)|||$ by using (23)-(24) and by applying the Gronwall's Lemma

$$|||\mathbb{P}(e)(T,\cdot)|||^2 \leq C(k,\gamma,\alpha,T)(\Delta x)^{2k}|U(T,\cdot)|^2_{k+1}.$$

Finally, the result follows from (20) and Lemma 5.1. □

6. Implementation of the numerical method

We conclude this paper by presenting some experimental results obtained using the numerical scheme (7). We consider Legendre polynomials $\{\phi_l^j\}_{l=0,\cdots,k-1}$ as a local orthogonal basis of the space $P^k(I_j)$ and we denote by $\{\phi_l\}_{l=0,\cdots,k-1}$ the orthogonal basis of the space $P^k(-1,1)$.

Therefore the numerical solution u for $x\in I_j$ in space V^k can be approximated by

$$u(t,x)=\sum_{l=0}^{k} c_l^j(t)\phi_l^j(x)=C_j^T(t)\Phi_j(x), \tag{25}$$

where $\Phi_j=[\phi_0^j,\cdots,\phi_k^j]^T$ and $C_j(t)=[c_0^j,c_1^j,\cdots,c_k^j]^T$ is the unknown to be determined from the numerical scheme (7).
From (7) with $v=\phi_p^j, p=0,\cdots,k$ we have

$$\begin{aligned}
&\sum_{l=0}^{k}\frac{d\,c_l^j(t)}{dt}\int_{I_j}\phi_l^j\phi_p^j(x)\,dx-\int_{I_j}f(u)(\phi_p^j)'(x)\,dx+\sum_{l=0}^{k}c_l^j(t)\int_{I_j}(\phi_l^j)'(\phi_p^j)'(x)\,dx\\
&+\hat f(u)\phi_p^j(x_{j+1}^-)-\hat f(u)\phi_p^j(x_j^+)-\hat u_x(x_{j+1})\phi_p^j(x_{j+1}^-)+\hat u_x(x_j)\phi_p^j(x_j^+)\\
&-\int_{I_j}\mathcal{J}[u]\left(\phi_p^j\right)'+\widehat{\mathcal{J}[u]}(x_{j+1})\phi_p^j(x_{j+1}^-)-\widehat{\mathcal{J}[u]}(x_j)\phi_p^j(x_j^+)=0.
\end{aligned} \tag{26}$$

Using the following Legendre properties

$$\begin{aligned}
&\phi_l^j(x_{j-1/2})=(-1)^l,\quad \phi_l^j(x_{j+1/2})=1,\\
&\int_{I_j}\phi_l^j\phi_p^j(x)\,dx=\frac{\Delta x}{2p+1}\delta_{lp}=\alpha_p\delta_{lp},
\end{aligned}$$

we obtain from (26) for all $p=0,\cdots,k$,

$$\begin{aligned}
&\alpha_p\frac{d\,c_p^j(t)}{dt}-\int_{I_j}f(u)(\phi_p^j)'(x)\,dx+\sum_{l=0}^{k-1}c_l^j(t)\int_{I_j}(\phi_l^j)'(\phi_p^j)'(x)\,dx-\int_{I_j}\mathcal{J}[u]\left(\phi_p^j\right)'\\
&+\hat f(u_{j+1/2}^-,u_{j+1/2}^+)-(-1)^p\hat f(u_{j-1/2}^-,u_{j-1/2}^+)-\hat u_x(x_{j+1/2})+\hat u_x(x_{j-1/2})(-1)^p\\
&+\widehat{\mathcal{J}[u]}(x_{j+1/2})-\widehat{\mathcal{J}[u]}(x_{j-1/2})(-1)^p=0.
\end{aligned}$$

Test 1: We first test the numerical convergence of the DDG method (7) for linear and quadratic elements with explicit third order Runge-Kutta time discretization. We also use quadrature rules to compute all integrals. We consider here the following problem:

$$\begin{cases}\frac{\partial u(t,x)}{\partial t}+c\,\frac{\partial u(t,x)}{\partial x}-\frac{\partial^2 u(t,x)}{\partial x^2}+\partial_x\mathcal{J}[u(t,\cdot)](x) &= g(t,x),\quad x\in(-1,1),t>0\\ u(0,x) &= u_0(x)\end{cases} \tag{27}$$

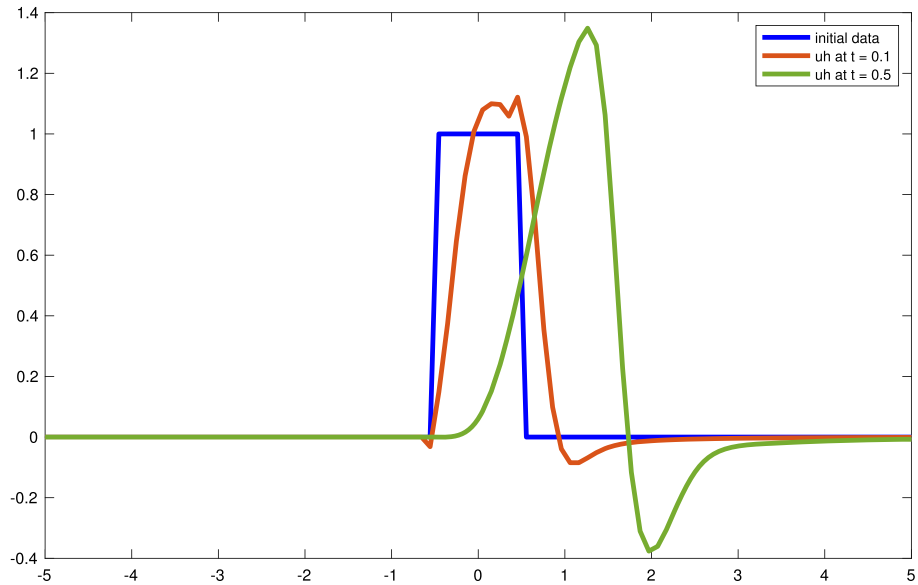

Fig. 2. Test 2: Numerical solutions of (28) with $\eta = 1, \varepsilon = 0.01$ and $N = 100$.

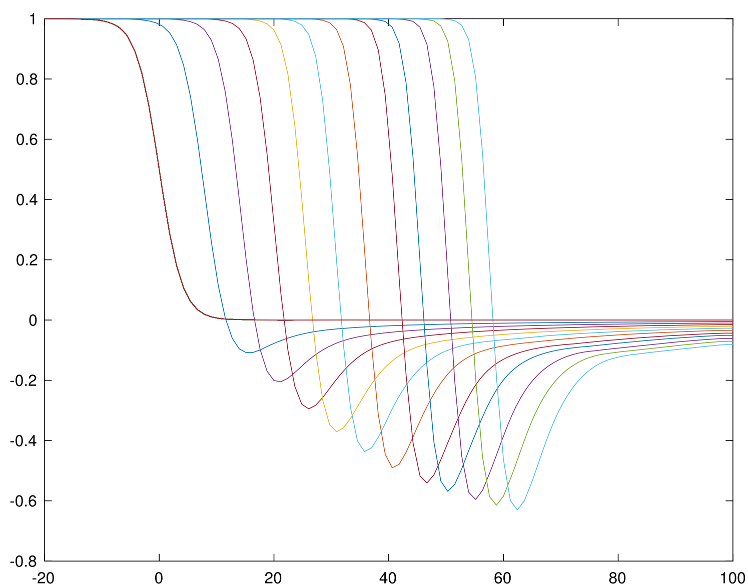

Fig. 3. Test 3: Numerical solutions for different times of (1) with the initial data (29).

with the initial condition $u_0(x) = e^{-50x^2}$, $c = 0.1$ and the corresponding forcing term $g(t,x)$ is given by

$$g(t,x) = e^{-t}\left(-u_0(x) + c\,u_0'(x) - u_0''(x) + \partial_x \mathcal{J}[u_0](x)\right).$$

In Fig. 1 we plot the logarithm of the error (in norm L^2) in function of the logarithm of the number of element N. The convergence numerical order is then given by the slope of the curve. We observe that the numerical rate of convergence is consistent with the theoretical result.

Test 2. In this test case we simulate the initial problem (1) where parameters are added to amplify the effects of the nonlocal operator on the diffusion term:

$$u_t + \left(\frac{u^2}{2} - \varepsilon u_x + \eta \mathcal{J}[u]\right)_x = 0, x \in (-5,5), t > 0 \tag{28}$$

with the following discontinuous initial data

$$u_0(x) = \begin{cases} 1 & |x| < 0.5, \\ 0 & \text{otherwise} \end{cases}$$

The numerical solutions for $t = 0.1$ and $t = 0.5$ of the problem (28) are shown in Fig. 2. From this figure it is clear that the maximum principle is not satisfied as it has been proved in [3] and that the nonlocal operator generates instabilities.

Test 3: Finally in this last case, we simulate the original problem (1) for two initial data:

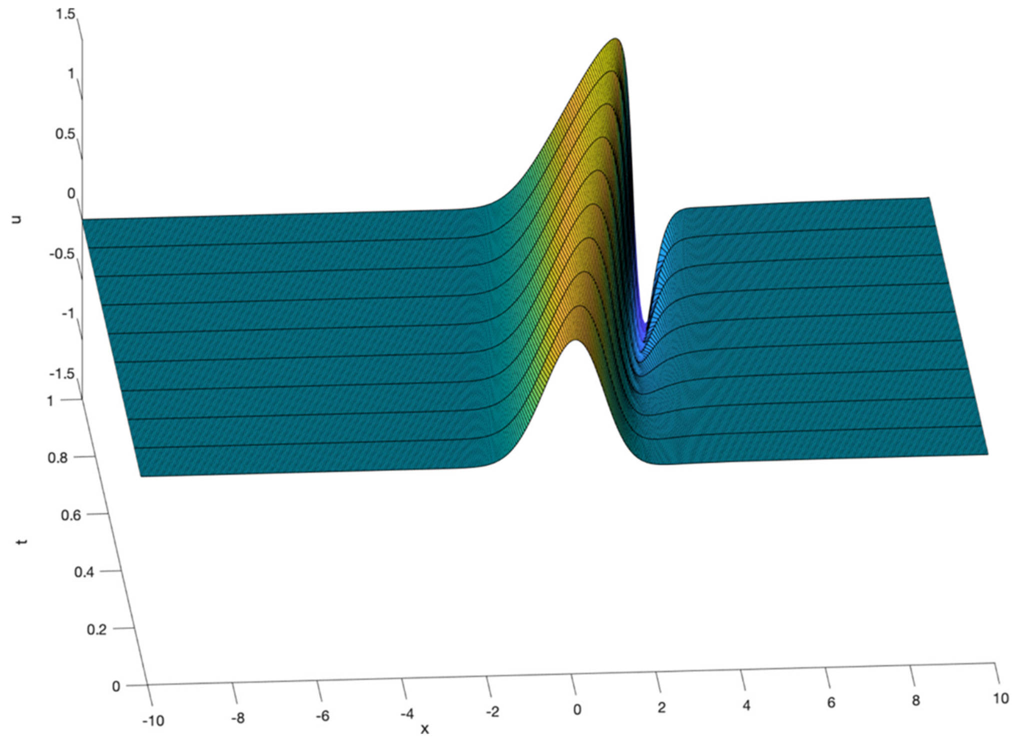

Fig. 4. Test 3. Numerical solutions of the morphodynamics model (1) with initial condition (30) for $t \in (0.1, 1)$.

- We first want to simulate the effect of the nonlocal term on the well-known traveling wave solution of the viscous Burgers equation. We then consider the following initial data:

$$u_0(x) = \frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{4}x) \tag{29}$$

  We expose in Fig. 3 the numerical solutions for different times. We note that the shape of the wave is not preserved due to the nonlocal operator.
- We now want to simulate the evolution of a dune morphodynamic by considering the following initial data:

$$u_0(x) = e^{-x^2} \tag{30}$$

We plot in Fig. 4 for different times the evolution of this initial data and we observe that with time, the dune deepens, which reflects the phenomenon of erosion related to the model of morphodynamics.

Data availability

No data was used for the research described in the article.

References

[1] T. Aboelenen, A direct discontinuous Galerkin method for fractional convection-diffusion and Schrödinger-type equations, Eur. Phys. J. Plus 133 (2018).
[2] T. Aboelenen, H. El-Hawary, A high-order nodal discontinuous Galerkin method for a linearized fractional Cahn–Hilliard equation, in: Advances in Fractional Differential Equations (IV): Time-Fractional PDEs, Comput. Math. Appl. 73 (2017) 1197–1217.
[3] N. Alibaud, P. Azerad, D. Isèbe, A non-monotone nonlocal conservation law for dune morphodynamics, Differ. Integral Equ. 23 (2010) 155–188.
[4] P. Azerad, A. Bouharguane, J.-F. Crouzet, Simultaneous denoising and enhancement of signals by a fractal conservation law, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 867–881.
[5] F. Betancourt, R. Bürger, K.H. Karlsen, E.M. Tory, On nonlocal conservation laws modelling sedimentation, Nonlinearity 24 (2011) 855.
[6] E. Bezchlebova, V. Dolejsi, M. Feistauer, Discontinuous Galerkin method for the solution of a transport level-set problem, Comput. Math. Appl. 72 (2016) 455–480.
[7] S. Blandin, P. Goatin, Well-posedness of a conservation law with non-local flux arising in traffic flow modeling, Numer. Math. 132 (2016) 217–241.
[8] A. Bouharguane, Finite element method for a space-fractional anti-diffusive equation, J. Comput. Appl. Math. 328 (2018) 497–507.
[9] A. Bouharguane, R. Carles, Splitting methods for the nonlocal Fowler equation, Math. Comput. 83 (2014).
[10] A. Bouharguane, N. Seloula, The local discontinuous Galerkin method for convection-diffusion-fractional anti-diffusion equations, Appl. Numer. Math. 148 (2020) 61–78.
[11] C. Chalons, P. Goatin, L.M. Villada, High-order numerical schemes for one-dimensional nonlocal conservation laws, SIAM J. Sci. Comput. 40 (2018) A288–A305.
[12] Y. Cheng, C.-W. Shu, A discontinuous Galerkin finite element method for time dependent partial differential equations with higher order derivatives, Math. Comput. 77 (2008) 699–730.
[13] P.G. Ciarlet, Finite Element Method for Elliptic Problems, Society for Industrial and Applied Mathematics, USA, 2002.
[14] S. Cifani, E.R. Jakobsen, K.H. Karlsen, The discontinuous Galerkin method for fractal conservation laws, IMA J. Numer. Anal. 31 (2010) 1090–1122.
[15] S. Cifani, E.R. Jakobsen, K.H. Karlsen, The discontinuous Galerkin method for fractional degenerate convection-diffusion equations, BIT Numer. Math. 51 (2011) 809–844.
[16] B. Cockburn, An Introduction to the Discontinuous Galerkin Method for Convection-Dominated Problems, Springer Berlin Heidelberg, Berlin, Heidelberg, 1998, pp. 150–268.
[17] B. Cockburn, C.-W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, SIAM J. Numer. Anal. 35 (1998) 2440–2463.
[18] B. Cockburn, C.-W. Shu, Runge–Kutta discontinuous Galerkin methods for convection-dominated problems, J. Sci. Comput. 16 (2001) 173–261.
[19] W. Deng, Finite element method for the space and time fractional Fokker–Planck equation, SIAM J. Numer. Anal. 47 (2009) 204–226.
[20] W.H. Deng, J.S. Hesthaven, Local discontinuous Galerkin methods for fractional diffusion equations, Modél. Math. Anal. Numér. 47 (2013) 1845–1864.

[21] V. Dolejsi, M. Feistaue, Discontinuous Galerkin Method—Analysis and Applications to Compressible Flow, Springer Series in Computational Mathematics, Springer, 2015.
[22] J. Droniou, A numerical method for fractal conservation laws, Math. Comput. 79 (2010) 95–124.
[23] A.C. Fowler, Dunes and drumlins, in: A. Provenzale, N. Balmforth (Eds.), Geomorphological Fluid Mechanics, vol. 211, Springer-Verlag, Berlin, 2001, pp. 430–454.
[24] Y. Huang, A. Oberman, Numerical methods for the fractional Laplacian: a finite difference-quadrature approach, SIAM J. Numer. Anal. 52 (2014) 3056–3084.
[25] K.K. Kouakou, P.-Y. Lagrée, Stability of an erodible bed in various shear flows, Eur. Phys. J. B 47 (2005) 115–125.
[26] C. Li, S. Zhao, Efficient numerical schemes for fractional water wave models, Comput. Math. Appl. 71 (2016) 238–254.
[27] H. Liu, Optimal error estimates of the direct discontinuous Galerkin method for convection-diffusion equations, Math. Comput. 84 (2015) 2263–2295.
[28] H. Liu, J. Yan, The direct discontinuous Galerkin (DDG) methods for diffusion problems, SIAM J. Numer. Anal. 47 (2009) 675–698.
[29] A.-M. Matache, C. Schwab, T.P. Wihler, Fast numerical solution of parabolic integrodifferential equations with applications in finance, SIAM J. Sci. Comput. 27 (2005) 369–393.
[30] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, J. Comput. Appl. Math. 172 (2004) 65–77.
[31] K. Mustapha, W. McLean, Superconvergence of a discontinuous Galerkin method for fractional diffusion and wave equations, SIAM J. Numer. Anal. 51 (2013) 491–515.
[32] H.M. Soner, Optimal control with state-space constraint. II, SIAM J. Control Optim. 24 (1986) 1110–1122.
[33] Q. Xu, J.S. Hesthaven, Discontinuous Galerkin method for fractional convection-diffusion equations, SIAM J. Numer. Anal. 52 (2014) 405–423.
[34] Y. Xu, C.-W. Shu, Error estimates of the semi-discrete local discontinuous Galerkin method for nonlinear convection–diffusion and kdv equations, Comput. Methods Appl. Mech. Eng. 196 (2007) 3805–3822.
[35] Q. Zhang, C. Shu, Error estimates to smooth solutions of Runge–Kutta discontinuous Galerkin method for symmetrizable systems of conservation laws, SIAM J. Numer. Anal. 44 (2006) 1703–1720.
[36] Y. Zheng, C. Li, Z. Zhao, A note on the finite element method for the space-fractional advection diffusion equation, Comput. Math. Appl. 59 (2010) 1718–1726.