DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **6**, Number **5**, October **2013**

doi:10.3934/dcdss.2013.6.1113

pp. **1113–1137**

L^p-THEORY FOR THE NAVIER-STOKES EQUATIONS WITH PRESSURE BOUNDARY CONDITIONS

Chérif Amrouche

Université de Pau et des Pays de l'Adour
LMA, Avenue de l'Université
64013 Pau cedex, France

Nour El Houda Seloula

Laboratoire de Mathématiques Nicolas Oresme BP 5186
UMR 6139 CNRS
Université de Caen Basse Normandie 14032 Cedex, France

Abstract. We consider the Navier-Stokes equations with pressure boundary conditions in the case of a bounded open set, connected of class $\mathcal{C}^{1,1}$ of $\mathbb{R}^3$. We prove existence of solution by using a fixed point theorem over the type-Oseen problem. This result was studied in [5] in the Hilbertian case. In our study we give the L^p-theory for $1 < p < \infty$.

1. **Introduction.** Let Ω be a bounded open set, connected of class $\mathcal{C}^{1,1}$ of $\mathbb{R}^3$ with boundary Γ. Let Γ_i, $0 \le i \le I$, denote the connected components of the boundary Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \Omega$. We do not assume that Ω is simply-connected but we suppose that there exist J connected open surfaces Σ_j, $1 \le j \le J$, called 'cuts', contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold. The boundary of each Σ_j is contained in Γ. The intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ is empty for $i \neq j$, and finally the open set $\Omega^\circ = \Omega \setminus \cup_{j=1}^J \Sigma_j$ is simply-connected.

We are interested to the study of solutions to the Navier-Stokes equations:

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \quad \text{and} \quad \pi = \pi_0 & \text{on } \Gamma, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n} \, d\boldsymbol{\sigma} = 0, \ i = 1, \dots, I. \end{cases} \tag{1}$$

This type of boundary conditions appears in a large number of physical situations, as for instance in case od pipelines, blood vessels, different hydraulic systems involving pumps. Our goal here is to prove the existence of weak solutions for small data in L^p-theory with $1 < p < \infty$. The existence of solutions in $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ can be directly obtained by the Lax-Milgram Lemma, without suppose the regularity of Ω. We can next, obtain weak solutions in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for $\frac{3}{2} < p < 3$ by using the fixed point technique over the type-Oseen equations. For a future work, the

2010 *Mathematics Subject Classification.* Primary: 35J25, 35J47, 35J50, 35J57; Secondary: 76D05, 76D07.

Key words and phrases. Stokes equations, Navier-Stokes equations, Oseen equations, pressure boundary conditions, inf-sup condition.

study of these last equations will be very useful for a more complete analysis of the Navier-Stokes equations, particularly for the existence of very weak solutions.

Before stating our results, we introduce some functions spaces. Let $\boldsymbol{L}^p(\Omega)$ denotes the usual vector-valued $\boldsymbol{L}^p-$space over Ω, $1 < p < \infty$. Let us define the spaces:

$$\boldsymbol{H}^{\,p}(\mathbf{curl},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\right\},$$

with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^p{}_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|^p{}_{\boldsymbol{L}^p(\Omega)}\right)^{\frac{1}{p}},$$

$$\boldsymbol{H}^{\,p}(\mathrm{div},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)\right\},$$

with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathrm{div},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$$

and $\boldsymbol{X}^{\,p}(\Omega) = \boldsymbol{H}^{\,p}(\mathbf{curl},\Omega) \cap \boldsymbol{H}^{\,p}(\mathrm{div},\Omega)$, equipped with the graph norm. As in the case of Hilbert spaces, we can prove that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^{\,p}(\mathbf{curl},\,\Omega)$, $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ and $\boldsymbol{X}^{\,p}(\Omega)$. We also define the subspaces:

$$\boldsymbol{H}^{\,p}_0(\mathbf{curl},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^{\,p}(\mathbf{curl},\Omega);\;\; \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\right\},$$

$$\boldsymbol{H}^{\,p}_0(\mathrm{div},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^{\,p}(\mathrm{div},\Omega);\;\; \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\right\},$$

$$\boldsymbol{X}^{\,p}_N(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{X}^{\,p}(\Omega);\, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\right\},\; \boldsymbol{X}^{\,p}_T(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{X}^{\,p}(\Omega); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\right\}$$

and $\boldsymbol{X}^{\,p}_0(\Omega) = \boldsymbol{X}^{\,p}_N(\Omega) \cap \boldsymbol{X}^{\,p}_T(\Omega)$. We introduce the spaces

$$\boldsymbol{H}^{\,1}_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{\,1}(\Omega);\;\; \mathrm{div}\,\boldsymbol{v} = 0 \text{ in}\,\Omega\}.$$

$$\boldsymbol{L}^s_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^{\,s}(\Omega);\;\; \mathrm{div}\,\boldsymbol{v} = 0 \text{ in}\,\Omega\}.$$

We note that the space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\boldsymbol{L}^s_\sigma(\Omega)$ (see [2, Lemma 7]).

Finally, we define the space

$$\boldsymbol{K}^{\,p}_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\; \mathrm{div}\,\boldsymbol{v} = 0,\; \mathbf{curl}\,\boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}.$$

which is of finite dimension and is spanned by the functions ∇q^N_i, $i = 1\dots, I$, where q^N_i is the unique solution in $W^{\,2,p}(\Omega)$ of the problem

$$\begin{cases} -\Delta q^N_i = 0 \qquad \text{in } \Omega, \\ q^N_i|_{\Gamma_0} = 0 \quad \text{and} \quad q^N_i|_{\Gamma_k} = \text{constant}, \;\; 1 \le k \le I, \\ \left\langle \partial_n\, q^N_i,\, 1\right\rangle_{\Gamma_k} = \delta_{i\,k},\;\; 1 \le k \le I, \quad \text{and} \quad \left\langle \partial_n\, q^N_i,\, 1\right\rangle_{\Gamma_0} = -1. \end{cases} \tag{2}$$

The work is organised as follows: in the next section, we present some preliminary results including the study of the existence of solution for the Stokes problem when the pressure and the tangential velocity are given on the boundary. In Section 3, we extend previous results for the Oseen problem inorder to obtain in Section 4 the result for the Navier-Stokes equations using a fixed point technique.

2. **The Stokes problem.** We first recall some results on the following Stokes problem

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \le i \le I, \end{cases} \tag{3}$$

that we can find in [3]. The first two points of the following Theorem concerns the existence of weak and strong solutions for the problem (3) when the data satisfy some compatibilty condition. In the third point, we treat the case where the pressure given by the previous points is equal to zero.

Theorem 2.1.
i) *Let* $\boldsymbol{f}$, $\boldsymbol{g}$, π_0 *with*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]', \quad \boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{\,1-1/p,p}(\Gamma), \tag{4}$$

satisfying the compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle\, \boldsymbol{f},\, \boldsymbol{v} \rangle_\Omega - \int_\Gamma \pi_0\, \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{s} = 0, \tag{5}$$

where $\langle \cdot, \cdot \rangle_\Omega = \langle \cdot, \cdot \rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)}$. *Then, the Stokes problem* (3) *has a unique solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times\, W^{\,1,p}(\Omega)$ *satisfying the estimate*

$$\|\, \boldsymbol{u} \,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\, \pi \,\|_{W^{\,1,p}(\Omega)} \le C \big(\|\, \boldsymbol{f} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'} \;+\; \|\boldsymbol{g} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \\ + \; \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)} \big).$$

ii) *Moreover, if* Ω *is of class* $\mathcal{C}^{\,2,1}$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, *then the solution* $(\boldsymbol{u},\, \pi)$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega)$ *and satisfies the estimate:*

$$\|\, \boldsymbol{u} \,\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\, \pi \,\|_{W^{\,1,p}(\Omega)} \le C \big(\|\, \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g} \times \boldsymbol{n}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)} \big).$$

iii) *Moreover, if* $\operatorname{div} \boldsymbol{f} = 0$ *in* Ω, $\pi_0 = 0$ *and* $\boldsymbol{g} \times \boldsymbol{n} = \mathbf{0}$ *on* Γ, *then* $\pi = 0$.

The folowing lemma gives some properties of normal traces of $\mathbf{curl}\, \boldsymbol{v}$ for some functions $\boldsymbol{v}$.

Lemma 2.2. *For any* $\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}\,; \Omega)$, *we have the relation:*

$$\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = \operatorname{div}_T (\boldsymbol{v} \times \boldsymbol{n}) \ \text{ in the sense of } W^{\,-1/p,p}(\Gamma), \tag{6}$$

where div_T *is the tangential divergence. If moreover,* $\boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, *then* $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} \in W^{\,1-1/p,p}(\Gamma)$ *with the estimate:*

$$\|\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}\|_{W^{\,1-1/p,p}(\Gamma)} \le C \|\boldsymbol{v} \times \boldsymbol{n}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}.$$

The above relation (5) is a necessary condition for the existence of solution for the Stokes problem (3). Now, our goal is to see what happens precisely, when the data do not satisfy the compatibility condition (5).

As will appear, the answer strongly depends on the following variant of the Stokes problem $(\mathcal{S}_N)$: Find functions $\boldsymbol{u}$, π and constants c_i for $i = 1, \ldots, I$, such that:

$$(\mathcal{S}'_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on}\, \Gamma, \\ \pi = \pi_0 \ \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + c_i & \text{on}\, \Gamma_{\mathrm{i}}, \ \ 1 \le i \le I \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \le i \le I, \end{cases}$$

situation that we can be found also in the paper [5]. We recall the following result proved in [3].

Theorem 2.3. *Let* $\boldsymbol{f}$, $\boldsymbol{g}$ *and* π_0 *such that*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]', \quad \boldsymbol{g}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/p,p}(\Gamma).$$

Then, the problem $(\mathcal{S}'_N)$ *has a unique solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,p}(\Omega)$ *and constants* $c_1,\ldots,c_I$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|_{W^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\big),$$

and where $c_1,\ldots,c_I$ *are given by*

$$c_i = \langle \boldsymbol{f},\, \nabla q_i^N\rangle_\Omega - \langle \pi_0,\, \nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma \tag{7}$$

(the brackets on Ω *are the duality product between* $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$ *and* $\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)$*). Moreover, if* Ω *is of class* $\mathcal{C}^{2,1}$, $\boldsymbol{f}\in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g}\times\boldsymbol{n}\in \boldsymbol{W}^{2-1/p,p}(\Gamma)$*, then* $\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)$.

Remark 1. Observe that if we suppose that the compatibility condition (5) is verified, we obtain $c_i = 0$ for all $i = 1,\ldots,I$. Then, we are reduced to solve the problem $(\mathcal{S}'_N)$ without the constant c_i and $(\mathcal{S}'_N)$ is anything other than $(\mathcal{S}_N)$.

The assumption on $\boldsymbol{f}$ in Theorem 2.3 can be weakened by considering the space defined for all $1 < r,\, p < \infty$:

$$\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega) = \{\boldsymbol{\varphi}\in\boldsymbol{L}^r(\Omega);\ \mathbf{curl}\,\boldsymbol{\varphi}\in\boldsymbol{L}^p(\Omega),\ \ \boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0}\ \text{on}\,\Gamma\},$$

which is a Banach space for the norm

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)} = \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}.$$

The proof of the following lemma is similar to that of ([2, Lemma 8]), although the functional spaces are changed.

Lemma 2.4. *The space* $\boldsymbol{\mathcal{D}}(\Omega)$ *is dense in* $\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)$

Proof. In a first step, we consider that Ω is strictly star-shaped with respect to one of its points which is taken as the origin. Then, for any $\boldsymbol{\varphi}\in\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)$, we take $\widetilde{\boldsymbol{\varphi}}$ its extension by zero to $\mathbb{R}^3$. Thus, $\widetilde{\boldsymbol{\varphi}}\in\boldsymbol{L}^r(\mathbb{R}^3)$ and as $\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}} = \widetilde{\mathbf{curl}\,\boldsymbol{\varphi}}\in\boldsymbol{L}^p(\mathbb{R}^3)$, then $\widetilde{\boldsymbol{\varphi}}\in\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\mathbb{R}^3)$. For $\theta < 1$, we define the functions:

$$\widetilde{\boldsymbol{\varphi}}_\theta(\boldsymbol{x}) = \widetilde{\boldsymbol{\varphi}}(\frac{\boldsymbol{x}}{\theta}), \quad \text{for a.e } \boldsymbol{x}\in\mathbb{R}^3.$$

Since $\operatorname{supp}\widetilde{\boldsymbol{\varphi}}_\theta \subset \theta\overline{\Omega}\subset\Omega$, the function $\widetilde{\boldsymbol{\varphi}}_\theta$ has a compact support in Ω. Moreover, $\widetilde{\boldsymbol{\varphi}}_\theta\in\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\mathbb{R}^3)$ and

$$\lim_{\theta\to1}\widetilde{\boldsymbol{\varphi}}_\theta = \widetilde{\boldsymbol{\varphi}}\ \text{ in }\ \boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\mathbb{R}^3).$$

The result is then proved by regularization. Let $\rho\in\mathcal{D}(\mathbb{R}^3)$, be a smooth $\mathcal{C}^\infty$ function with compact support, such that $\rho\geq 0$, $\int_{\mathbb{R}^3}\rho(x)\,\mathrm{d}x = 1$. For $\varepsilon > 0$, let ρ_ε denote the function $x\longmapsto(\frac{1}{\varepsilon^3})\rho(\frac{x}{\varepsilon})$. As $\varepsilon\to0$, ρ_ε converges in the distribution sence to the Dirac distribution. As a consequence, $\rho_\epsilon * \widetilde{\boldsymbol{\varphi}}_\theta|_\Omega$ belongs to $\boldsymbol{\mathcal{D}}(\Omega)$ and

$$\lim_{\epsilon\to0}\lim_{\theta\to1}\rho_\epsilon * \widetilde{\boldsymbol{\varphi}}_\theta = \widetilde{\boldsymbol{\varphi}}\ \text{ in }\ \boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\mathbb{R}^3).$$

The result follows because $\boldsymbol{\varphi}$ is the limit in $\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)$ of the restriction of the functions $\rho_\epsilon * \widetilde{\boldsymbol{\varphi}}_\theta$ to Ω. In the case where Ω is not star-shaped, we have to recover Ω which is Lipschitz by a finite number of star open sets. □

The following lemma is classical

Lemma 2.5. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)]'$. *Then, there exist* $\boldsymbol{F} \in \boldsymbol{L}^{r'}(\Omega)$ *and* $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ *such that:*

$$\boldsymbol{f} = \boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi}. \tag{8}$$

Conversely, if $\boldsymbol{f}$ *satisfies* (8), *then* $\boldsymbol{f} \in [\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)]'$.

Proof. We set $\boldsymbol{E} = \boldsymbol{L}^r(\Omega) \times \boldsymbol{L}^p(\Omega)$ endowed whith the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{E}} = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

The application $T:\ \boldsymbol{v} \in \boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega) \mapsto (\boldsymbol{v},\,\mathbf{curl}\boldsymbol{v}) \in \boldsymbol{E}$ is an isometry from $\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)$ into $\boldsymbol{E}$. Hence, with each $\boldsymbol{g} \in [\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)]'$, we associate the element $\boldsymbol{g}^* \in (\mathcal{R}(T))'$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega), \quad \langle\, \boldsymbol{g},\, \boldsymbol{v}\,\rangle_{[\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)]'\times \boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)} = \langle\, \boldsymbol{g}^*,\, T\boldsymbol{v}\,\rangle_{(\mathcal{R}(T))'\times\mathcal{R}(T)}.$$

By the Hahn-Banach Theorem , $\boldsymbol{g}^*$ can be extended in $\boldsymbol{L}^{r'}(\Omega) \times \boldsymbol{L}^{p'}(\Omega)$ to an element called $(\boldsymbol{F},\,\boldsymbol{\psi})$. We deduce that:

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega), \langle\, \boldsymbol{g},\, \boldsymbol{v}\,\rangle_{[\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)]'\times \boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)} = \int_\Omega (\boldsymbol{F}\cdot\boldsymbol{v} + \boldsymbol{\psi}\cdot\mathbf{curl}\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}.$$

So, $\boldsymbol{g}$ is equal to $\boldsymbol{F}+\mathbf{curl}\,\boldsymbol{\psi}$ in Ω. It is easy to verify that the reciprocal holds. □

Theorem 2.6. *We assume that* Ω *is of class* $\mathcal{C}^{2,1}$. *Let* $\boldsymbol{f}$, $\boldsymbol{g}$ *and* π_0 *such that*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]', \quad \boldsymbol{g}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/r,r}(\Gamma),$$

with $r \le p$ *and* $\frac{1}{r} \le \frac{1}{p} + \frac{1}{3}$. *Then, the problem* $(\mathcal{S}'_N)$ *has a unique solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,r}(\Omega)$ *and constants* $c_1,\ldots,c_I$ *satisfying the estimate:*

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)} \quad &\le \quad C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \\ &+ \quad \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big),\end{aligned}$$

and $c_1,\ldots,c_I$ *are given by* (7), *where we replace the duality brackets on* Ω *by* $\langle\,\cdot,\cdot\,\rangle_\Omega = \langle\,\cdot,\cdot\,\rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'\times\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)}$.

Proof. Due to the characterization (8), we can write $\boldsymbol{f}$ as $\boldsymbol{f} = \boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi}$, where $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$. By [3, Proposition 5.1], the following problem:

$$\begin{cases}-\Delta\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{\psi} \ \text{ and } \ \mathrm{div}\,\boldsymbol{w} = 0 & \text{in}\,\Omega,\\ \boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} & \text{on}\,\Gamma,\\ \langle \boldsymbol{w}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \quad 1\le i\le I,\end{cases}$$

has a unique solution $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ (note that for any $1 \le i \le I$, $\langle \mathbf{curl}\,\boldsymbol{\psi},\ \nabla q_i^N\rangle_\Omega = 0$). Now, by Theorem 2.3, the following problem

$$\begin{cases}-\Delta\,\boldsymbol{z} + \nabla\pi = \boldsymbol{F} \ \text{ and } \ \mathrm{div}\,\boldsymbol{z} = 0 & \text{in}\,\Omega,\\ \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma,\\ \pi = \pi_0 \ \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + c_i & \text{on}\,\Gamma_i, \ \ 1\le i\le I,\\ \langle \boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \quad 1\le i\le I,\end{cases}$$

has a unique solution $(\boldsymbol{z},\,\pi,\,\boldsymbol{c}) \in \boldsymbol{W}^{2,r}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$, where c_i is given by (7). Observe that, since $\frac{1}{r} \le \frac{1}{p} + \frac{1}{3}$, $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$. Setting $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{z}$, then $(\boldsymbol{u},\,\pi,\,\boldsymbol{c})$ is the unique solution of the problem $(\mathcal{S}'_N)$. □

3. **The type-Oseen problem.** In this section, we are interested to the study of the following type-Oseen problem:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = \pi_0 + c_i & \text{on } \Gamma_i,\ i = 0, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I. & \end{cases} \tag{9}$$

First, we will study the existence of weak solutions $(\boldsymbol{u},\, \pi) \in \boldsymbol{H}^{\,1}(\Omega) \times L^2(\Omega)$ with $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\boldsymbol{f} \in [\boldsymbol{H}_0^{\,2}(\mathbf{curl},\, \Omega)]'$. Next, we will give the good conditions to ensure the existence of strong solutions $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{\,2,p}(\Omega) \times W^{\,1,p}(\Omega)$ for $p \geq 6/5$. Remark that we can replace in this section the term $\mathbf{curl}\, \boldsymbol{a}$ by a general given vector $\boldsymbol{d}$ in $\boldsymbol{L}^{3/2}(\Omega)$.

3.1. Weak solutions in $\boldsymbol{H}^{\,1}(\Omega) \times L^2(\Omega)$.

We introduce the space:

$$\boldsymbol{V}_N = \Big\{ \boldsymbol{v} \in H^1(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega,\ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma \ \text{ and } \ \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n} = 0,\ 1 \leq i \leq I \Big\}.$$

Let us give the following lemma concerning a Poincaré type inequality which is a consequence of [3, Corollary 3.2] (see also [5]).

Lemma 3.1. *The space $\boldsymbol{V}_N$ is a Hilbertian space and the semi norm:*

$$\boldsymbol{v} \mapsto \Big(\int_\Omega |\mathbf{curl}\, \boldsymbol{v}|^2 \Big)^{1/2} \tag{10}$$

is a norm on $\boldsymbol{V}_N$ equivalent to the full norm of $\boldsymbol{H}^1(\Omega)$.

Before establishing the result of existence of a weak solution for the problem (9), we will see in what functional space it is reasonable to find the pressure π appearing in (9), knowing that we are first interesting to velocity fields in $\boldsymbol{u} \in \boldsymbol{H}^{\,1}(\Omega)$ with $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$. With a such vector $\boldsymbol{u}$, we have $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^{\,6/5}(\Omega) \hookrightarrow \boldsymbol{H}^{\,-1}(\Omega)$. Since $\Delta\, \boldsymbol{u} \in \boldsymbol{H}^{\,-1}(\Omega)$, we deduce from the first equation in (9) that $\nabla\, \pi \in \boldsymbol{H}^{\,-1}(\Omega)$. According to [1, Proposition 2.10], the pressure π belongs to $L^2(\Omega)$. Furtheremore,

$$-\Delta\, \pi = \operatorname{div} \boldsymbol{f} - \operatorname{div}\, (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \qquad \text{in } \Omega,$$

so that $\Delta\, \pi \in \boldsymbol{W}^{\,-1,6/5}(\Omega)$. Using the same arguments as in [2, Lemma 2], we prove that the trace of π on Γ belongs to $H^{\,-1/2}(\Gamma)$ so that we must assume that $\pi_0 \in H^{\,-1/2}(\Gamma)$.

We have the following result.

Theorem 3.2. *Let $\boldsymbol{f} \in \boldsymbol{L}^{\,6/5}(\Omega)$, $\pi_0 \in H^{\,-1/2}(\Gamma)$ and $\boldsymbol{a} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{\,3/2}(\Omega)$. Then, the problem:*

$$\textit{Find } (\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{V}_N \times L^2(\Omega) \times \mathbb{R}^{I+1} \textit{ satisfying (9) with } \langle \pi,\, 1\rangle_\Gamma = 0 \tag{11}$$

is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{V}_N$ such that

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}_N, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\, \boldsymbol{x} - \langle \pi_0,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma \tag{12}$$

and find constants $c_0, \dots, c_I$ *satisfying* $\sum_{i=0}^{I} c_i \,\mathrm{mes}\,\Gamma_i + \langle \pi_0,\, 1\rangle_\Gamma = 0$ *and such that for any* $i = 1, \dots, I$*:*

$$c_i - c_0 = \int_\Omega \boldsymbol{f}\cdot \nabla\, q_i^N \,\mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N \,\mathrm{d}\boldsymbol{x} - \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n}\rangle_\Gamma. \qquad (13)$$

Proof. **i)** Let $(\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$, with $\langle \pi,\, 1\rangle_\Gamma = 0$, solution of the problem (9). It is clear that $\boldsymbol{u} \in \boldsymbol{V}_N$ and that $\sum_{i=0}^{I} c_i \,\mathrm{mes}\,\Gamma_i + \langle \pi_0,\, 1\rangle_\Gamma = 0$. Let us check that $\boldsymbol{u}$ satisfies (12). Indeed, multiplying the first equation of (9) by $\boldsymbol{v} \in \boldsymbol{V}_N$, integrating by parts in Ω, we obtain

$$\int_\Omega (-\Delta\, \boldsymbol{u} + \nabla\, \pi) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \qquad (14)$$

where we observe that $-\Delta\, \boldsymbol{u} + \nabla\, \pi$ belongs to $\boldsymbol{L}^{6/5}(\Omega)$. But $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is dense in the space

$$\boldsymbol{\mathcal{M}} = \big\{(\boldsymbol{u},\, \pi) \in \boldsymbol{H}^1_\sigma(\Omega) \times L^2(\Omega);\;\; -\Delta\, \boldsymbol{u} + \nabla\, \pi \in \boldsymbol{L}^{6/5}(\Omega)\big\}$$

(see [3, Lemma 5.5] for a similar proof) and we have the following Green formula: For any $(\boldsymbol{u},\, \pi) \in \boldsymbol{\mathcal{M}}$ and $\boldsymbol{\varphi} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ:

$$\int_\Omega (-\Delta\, \boldsymbol{u} + \nabla\, \pi) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} + \langle \pi,\, \boldsymbol{\varphi}\cdot \boldsymbol{n}\rangle_\Gamma. \qquad (15)$$

Taking into account the boundary condition that verify the pressure π on Γ, we have:

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}_N, \quad \langle \pi,\, \boldsymbol{v}\cdot \boldsymbol{n}\rangle_\Gamma = \langle \pi_0,\, \boldsymbol{v}\cdot \boldsymbol{n}\rangle_\Gamma, \qquad (16)$$

which implies that $\boldsymbol{u}$ satisfies (12). It remains to prove the relation (13).

Now, let $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and set

$$\boldsymbol{v}_0 = \boldsymbol{v} - \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v}\cdot \boldsymbol{n}\Big)\, \nabla\, q_i^N. \qquad (17)$$

Observe that $\boldsymbol{v}_0$ belongs to $\boldsymbol{V}_N$. Multiplying the first equation of (9) by $\boldsymbol{v}$, integrating by parts in Ω and using the relation (12) with the test function $\boldsymbol{v}_0$, we obtain

$$\begin{aligned} - &\sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v}\cdot \boldsymbol{n}\Big) \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N \,\mathrm{d}\boldsymbol{x} + \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v}\cdot \boldsymbol{n}\Big) \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N \,\mathrm{d}\boldsymbol{x} \\ = &\sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v}\cdot \boldsymbol{n}\Big)\Big((c_i - c_0) + \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n}\rangle_\Gamma\Big). \end{aligned}$$

Testing with $\boldsymbol{v} = \nabla\, q_j^N$, we deduce the required relation (13).

ii) Reciprocally, let $\boldsymbol{u} \in \boldsymbol{V}_N$ a solution of (12) and $c_0, \dots, c_I$ constants satisfying the condition $\sum_{i=0}^{I} c_i \,\mathrm{mes}\,\Gamma_i + \langle \pi_0,\, 1\rangle_\Gamma = 0$ and (13). To prove the first equation of the problem (9), we take $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ as a test function in (12) and we use the De Rham's Theorem. Moreover, since $\Delta\, \boldsymbol{u} \in \boldsymbol{H}^{-1}(\Omega)$ and $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^{6/5}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$, we deduce that $\nabla\, \pi \in \boldsymbol{H}^{-1}(\Omega)$. Due to [1], π belongs to $L^2(\Omega)$. Now, applying divergence operator to the first equations of problem (9), we obtain:

$$\Delta\, \pi = \mathrm{div}\, \boldsymbol{f} - \mathrm{div}\, (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \in W^{-1,\,6/5}(\Omega). \qquad (18)$$

Since $\pi \in L^2(\Omega)$, we can prove that the trace of π on Γ belongs to $H^{-1/2}(\Gamma)$ (see [2]). Then, since π is defined up to an additive constant, we can choose this constant so that $\langle \pi,\, 1\rangle_\Gamma = 0$.

It remains to prove the boundary condition on the pressure. Let $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. Multiplying the first equation of (9) by $\boldsymbol{v}$, integrating by parts in Ω and using the decomposition (17), we obtain as before:

$$\begin{aligned}
&\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}_0 \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v}_0 \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_0 \,\mathrm{d}\boldsymbol{x} + \langle \pi,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma \\
= &\sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla q_i^N \,\mathrm{d}\boldsymbol{x} - \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \int_\Omega \boldsymbol{f} \cdot \nabla q_i^N \,\mathrm{d}\boldsymbol{x} + \\
+ &\sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \langle \pi,\, \nabla q_i^N \cdot \boldsymbol{n}\rangle_\Gamma. \qquad (19)
\end{aligned}$$

Particularly, if $\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0$ for any $i = 1, \dots, I$, we deduce from (12) that:

$$\langle \pi,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma = \langle \pi_0,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma. \qquad (20)$$

Using again (12), the decomposition (17) and (20), the relation (19) becomes:

$$\begin{aligned}
\langle \pi,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma \;=\; &\langle \pi_0,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \int_\Omega \boldsymbol{f} \cdot \nabla q_i^N \,\mathrm{d}\boldsymbol{x} - \\
- &\sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla q_i^N \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

As a consequence, from the relation (13) and the fact that $\sum_{i=0}^{I} \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0$ we obtain:

$$\langle \pi,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma \;=\; \langle \pi_0,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma + \sum_{i=0}^{I} \langle c_i,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\Gamma_i} = \langle \pi_0 + c,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma, \qquad (21)$$

with $c = c_i$ on Γ_i for any $i = 0, \dots, I$.

Finally, let $\mu \in H^{1/2}(\Gamma_j)$ for any fixed j with $1 \le j \le I$ and we set

$$\alpha_j = \int_{\Gamma_j} \mu.$$

We know that there exists a vector $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ such that

$$\boldsymbol{v} = \begin{cases} \mu \boldsymbol{n} & \text{on } \Gamma_j, \\ \alpha_j \nabla q_j^N & \text{on } \Gamma - \Gamma_j. \end{cases} \qquad (22)$$

It is clear that $\boldsymbol{v}$ satisfies: $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and $\int_\Gamma \boldsymbol{v} \cdot \boldsymbol{n} = 0$. Hence, from (21), we can write:

$$\begin{aligned}
\langle \pi - \pi_0,\, \mu\rangle_{\Gamma_j} + \alpha_j \langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma - \Gamma_j} \;&=\; c_j \int_{\Gamma_j} \mu - c_0\, \alpha_j \\
&=\; (c_j - c_0)\alpha_j
\end{aligned}$$

Using again (21) with $\boldsymbol{v} = \nabla q_j^N$, we can write:

$$\begin{aligned}\langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma-\Gamma_j} &= \langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma} - \langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma_j} \\ &= c_j - c_0 - \langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma_j}.\end{aligned}$$

As a consequence,

$$\langle \pi - \pi_0,\, \mu\rangle_{\Gamma_j} = \alpha_j \langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma_j} = \Big(\int_{\Gamma_j} \mu\Big)\langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma_j},$$

and then for any $1 \leq j \leq I$, we deduce that $\pi - \pi_0$ is constant on Γ_j:

$$\pi - \pi_0 = d_j \quad \text{on } \Gamma_j \qquad \text{with} \quad d_j = \langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma_j}. \tag{23}$$

Next, as above, there exists a vector $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ such that

$$\boldsymbol{v} = \begin{cases} \mu \boldsymbol{n} & \text{on } \Gamma_0, \\ \beta_j \nabla q_j^N & \text{on } \Gamma - \Gamma_0, \end{cases}$$

with arbitrary fixed j satisfying $1 \leq j \leq I$ and where $\mu \in H^{1/2}(\Gamma_0)$ and

$$\beta_j = -\int_{\Gamma_0} \mu.$$

With the same arguments we obtain:

$$\pi - \pi_0 = d_0 \quad \text{on } \Gamma_0 \qquad \text{with} \quad d_0 = -\langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_{\Gamma_0}. \tag{24}$$

Because $\langle \pi,\, 1\rangle_\Gamma = 0$, it is easy to verify the relation:

$$\sum_{j=0}^{I} d_j \,\mathrm{mes}(\Gamma_j) + \langle \pi_0,\, 1\rangle_\Gamma = 0.$$

Using (21) with $\boldsymbol{v} = \nabla q_j^N$, we deduce that

$$d_j - d_0 = \langle \pi - \pi_0,\, \nabla q_j^N \cdot \boldsymbol{n}\rangle_\Gamma = c_j - c_0.$$

This finishes the proof, because the system (13) with the condition

$$\sum_{j=0}^{I} c_j \,\mathrm{mes}(\Gamma_j) + \langle \pi_0,\, 1\rangle_\Gamma = 0$$

admits a unique solution $\boldsymbol{c}$. □

Theorem 3.3. *Let $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\pi_0 \in H^{-1/2}(\Gamma)$, then the problem (9) has a unique solution $(\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times \mathbb{R}^{I+1}$ with $\langle \pi,\, 1\rangle_\Gamma = 0$ and we have the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{25}$$

$$\|\pi\|_{L^2(\Omega)} \leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{26}$$

where $\boldsymbol{c} = (c_0, \ldots, c_I)$. Moreover, if $\pi_0 \in W^{1/6,6/5}(\Gamma)$ and Ω is $\mathcal{C}^{2,1}$, then $\boldsymbol{u} \in \boldsymbol{W}^{2,6/5}(\Omega)$ and $\pi \in W^{1,6/5}(\Omega)$.

Proof. **i)** We suppose that $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\pi_0 \in H^{-1/2}(\Gamma)$. We know due to Theorem 3.2 that the problem (9) is equivalent to (12)-(13). Let $a(\cdot,\cdot) : \boldsymbol{V}_N \times \boldsymbol{V}_N \to \mathbb{R}$ be the following bilinear continuous form:

$$\forall \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{V}_N, \quad a(\boldsymbol{u},\, \boldsymbol{v}) = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}. \tag{27}$$

Using Lemma 3.1 and the fact that for any $\boldsymbol{v} \in \boldsymbol{V}_N$ we have:

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{v}) \cdot \boldsymbol{v} = 0,$$

we deduce that the form $a(\cdot,\cdot)$ is coercive. Hence, by the Lax-Milgram theorem, problem (9) has a unique solution $(\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$ satisfying the estimate (25). The estimate (26) can be obtained by using the fact that the pressure π verifies:

$$-\Delta \pi = \operatorname{div} \boldsymbol{f} + \operatorname{div} (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \text{ in } \Omega \quad \text{and} \quad \pi = \pi_0 + c_i \text{ on } \Gamma_i, \quad 0 \le i \le I. \tag{28}$$

ii) Now, we suppose that $\pi_0 \in W^{1/6,6/5}(\Gamma)$. Let $(\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$ the solution of (9) given by step i). Since the pressure π verifies (28), we deduce directly that $\pi \in W^{1,6/5}(\Omega)$. The velocity $\boldsymbol{u}$ is then a solution of the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \ i = 1, \ldots, I, \end{cases}$$

with $\boldsymbol{F} = \boldsymbol{f} - \nabla \pi - \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^{6/5}(\Omega)$. We deduce by Theorem 2.1 point (iii) that $\boldsymbol{u} \in \boldsymbol{W}^{2,6/5}(\Omega)$. □

Remark 2. Even if the pressure π change in $\pi - c_0$, the system (9) is equivalent to the following type-Oseen problem:

$$(\mathcal{OS}_N) \qquad \begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi = \pi_0 + \alpha_i, \ i = 1, \ldots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \ i = 1, \ldots, I, \end{cases}$$

where the unknowns constants satisfy for any $i = 1, \ldots, I$:

$$\alpha_i = \int_\Omega \boldsymbol{f} \cdot \nabla q_i^N\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla q_i^N\, \mathrm{d}\boldsymbol{x} - \langle \pi_0,\, \nabla q_i^N \cdot \boldsymbol{n} \rangle_\Gamma.$$

But, it is clear that the new pressure π does not satisfy the condition $\langle \pi,\, 1 \rangle_\Gamma = 0$.

Remark 3. If we suppose that $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\, \Omega)]'$, $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\pi_0 \in H^{-1/2}(\Gamma)$, then the problems (11) and (12)-(13) are again equivalent. The proof is similar to that of Theorem 3.2 with the difference that we use here the duality brackets between $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\, \Omega)]'$ and $\boldsymbol{H}_0^{6,2}(\mathbf{curl},\, \Omega)$ in place of the integral on Ω in the right hand side of (12) and the density of $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ in the space

$$\boldsymbol{\mathcal{M}} = \big\{ (\boldsymbol{u},\, \pi) \in \boldsymbol{H}_\sigma^1(\Omega) \times L^2(\Omega); \ -\Delta\, \boldsymbol{u} + \nabla \pi \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\, \Omega)]' \big\}.$$

It is easy now to extend Theorem 3.3 to the case where $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\, \Omega)]'$, the divergence operator does not vanish and the case of nonhomogeneous boundary conditions.

Theorem 3.4. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$, $\chi \in W^{1,6/5}(\Omega)$, $\pi_0 \in H^{-1/2}(\Gamma)$ *and* $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)$. *Then the problem*

$$\begin{cases} -\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi = \pi_0 + \alpha_i,\ i = 1, \ldots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{29}$$

has a unique solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ *verifying the estimate:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \quad &\leq \quad C\Big(\|\,\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big) \times \\ &\times \quad \big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\,\boldsymbol{g}\,\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big), \end{aligned} \tag{30}$$

$$\begin{aligned} \|\pi\|_{L^2(\Omega)} \quad &\leq C\big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\Big(\|\,\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \\ &+ \big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big) \times \big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\,\boldsymbol{g}\,\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big), \end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$. *Moreover, if* $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\pi_0 \in W^{1/6,6/5}(\Gamma)$, $\boldsymbol{g} \in \boldsymbol{W}^{7/6,6/5}(\Gamma)$ *and* Ω *is* $\mathcal{C}^{2,1}$, *then* $\boldsymbol{u} \in \boldsymbol{W}^{2,6/5}(\Omega)$ *and* $\pi \in W^{1,6/5}(\Omega)$.

Proof. **i)** Let $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)$. We know that there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{w} = \boldsymbol{g}$ on Γ and verifying:

$$\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{31}$$

Let $\theta \in H^2(\Omega)$ the unique solution of the problem:

$$\Delta\,\theta = \operatorname{div}\boldsymbol{w} - \chi \quad \text{in}\,\Omega \quad \text{and} \quad \theta = 0 \quad \text{on}\,\Gamma, \tag{32}$$

and we set $\boldsymbol{u}_0 = -\nabla\,\theta + \boldsymbol{w}$. Then $\boldsymbol{u}_0 \in \boldsymbol{H}^{\,1}(\Omega)$ and satisfies:

$$\operatorname{div}\boldsymbol{u}_0 = \chi \text{ in } \Omega \text{ and } \boldsymbol{u}_0 \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \text{ on } \Gamma,$$

with

$$\|\boldsymbol{u}_0\|_{\boldsymbol{H}^{\,1}(\Omega)} \leq C\big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big). \tag{33}$$

We know by Remark 3 that there exists a unique $(\boldsymbol{z},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{H}^{\,1}(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ solution of

$$\begin{cases} -\Delta\,\boldsymbol{z} + \mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{z} + \nabla\,\pi = \boldsymbol{F}_0 \quad \text{and} \quad \operatorname{div}\boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \pi = \pi_0 \ \text{ on } \Gamma_0, \ \ \pi = \pi_0 + \alpha_i,\ i = 1, \ldots, I & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{z}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0, \end{cases} \tag{34}$$

where $\boldsymbol{F}_0 = \boldsymbol{f} + \Delta\,\boldsymbol{u}_0 - \mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u}_0$. Observe that $\Delta\,\boldsymbol{u}_0 = \nabla\,\chi - \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}_0) \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$ which implies that $\boldsymbol{F}_0 \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$. Moreover, using the estimate (33), $\boldsymbol{z}$ satisfies the estimate:

$$\begin{aligned} \|\boldsymbol{z}\|_{\boldsymbol{H}^{\,1}(\Omega)} \quad &\leq \quad C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big) \times \\ &\times \quad \big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big). \end{aligned} \tag{35}$$

In addition, using again (33), the pressure π satisfies the estimate:

$$\begin{aligned}\|\pi\|_{L^2(\Omega)} \quad &\leq \quad C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'}+\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\times \\ &\times \quad \big(\|\chi\|_{W^{1,6/5}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)+\|\pi_0\|_{H^{-1/2}(\Gamma)}\Big).\end{aligned}$$

Finally, the pair of functions $(\boldsymbol{u},\,\pi)=(\boldsymbol{z}+\boldsymbol{u}_0),\,\pi)$ is the required solution.
ii) Now, we suppose that $\boldsymbol{f}\in\boldsymbol{L}^{6/5}(\Omega)$, $\pi_0\in W^{1/6,6/5}(\Gamma)$ and $\boldsymbol{g}\in\boldsymbol{W}^{7/6,6/5}(\Gamma)$. Let $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha})\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times\mathbb{R}^I$ the solution of (29) given by step i). Since the pressure π verifies

$$-\Delta\,\pi=\operatorname{div}\boldsymbol{f}+\Delta\,\chi-\operatorname{div}\left(\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}\right)\;\text{ in }\Omega,$$

where the right hand side belongs to $\boldsymbol{W}^{-1,6/5}(\Omega)$, then $\pi\in W^{1,6/5}(\Omega)$. The velocity $\boldsymbol{u}$ is then a solution of the following problem:

$$\begin{cases}-\Delta\,\boldsymbol{u}=\boldsymbol{F}\quad\text{and}\quad\operatorname{div}\boldsymbol{u}=\chi & \text{in }\Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{g}\times\boldsymbol{n} & \text{on }\Gamma,\\ \int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma}=0,\;i=1,\ldots,I,\end{cases}$$

with $\boldsymbol{F}=\boldsymbol{f}-\nabla\,\pi-\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}\in\boldsymbol{L}^{6/5}(\Omega)$. We set $\boldsymbol{z}=\mathbf{curl}\,\boldsymbol{u}$. Then the function $\boldsymbol{z}$ satisfies: $\boldsymbol{z}\in\boldsymbol{L}^{6/5}(\Omega)$, $\operatorname{div}\boldsymbol{z}=0$ in Ω and $\mathbf{curl}\,\boldsymbol{z}=\boldsymbol{F}+\nabla\,\chi\in\boldsymbol{L}^{6/5}(\Omega)$. Moreover, since $\boldsymbol{u}\times\boldsymbol{n}\in\boldsymbol{W}^{7/6,6/5}(\Gamma)$, applying Lemma 2.2 we obtain that $\boldsymbol{z}\cdot\boldsymbol{n}\in W^{1/6,6/5}(\Gamma)$. So, from [3], we deduce that $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,6/5}(\Omega)$ and then $\boldsymbol{u}\in\boldsymbol{W}^{2,6/5}(\Omega)$. □

3.2. **Strong solutions.** In the rest of this paper, we suppose that Ω is $\mathcal{C}^{2,1}$. In this subsection, we are interested in the study of strong solutions for the system $(\mathcal{OS}_N)$. When $p<\frac{3}{2}$, because the embedding $W^{2,p}(\Omega)\hookrightarrow W^{1,p*}(\Omega)$, the term $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}\in L^p(\Omega)$ and we can use the regularity results on the Stokes problem. But this is not more the case when $p\geq\frac{3}{2}$ and that $\mathbf{curl}\,\boldsymbol{a}$ belongs only to $\boldsymbol{L}^{3/2}(\Omega)$. We give in the following theorem the good conditions to ensure the existence of strong solutions.

Theorem 3.5. *Let $p\geq 6/5$,*

$$\boldsymbol{f}\in\boldsymbol{L}^p(\Omega),\;\;\pi_0\in W^{1-1/p,p}(\Gamma),\;\;\mathbf{curl}\,\boldsymbol{a}\in\boldsymbol{L}^s(\Omega)$$

with

$$s=\frac{3}{2}\;\text{ if }\;p<\frac{3}{2},\qquad s=p\;\text{ if }\;p>\frac{3}{2},\qquad s=\frac{3}{2}+\varepsilon\;\text{ if }\;p=\frac{3}{2},$$

for some arbitrary $\varepsilon>0$. Then, the solution $(\boldsymbol{u},\,\pi)$ given by Theorem 3.3 belongs to $\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|\pi\|_{W^{1,p}(\Omega)}\leq C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}+\|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\big).\quad(36)$$

Proof. We know by Theorem 3.3 that the solution $(\boldsymbol{u},\,\pi)$ belongs to $\boldsymbol{W}^{2,6/5}(\Omega)\times W^{1,6/5}(\Omega)$. We set $\boldsymbol{b}=\mathbf{curl}\,\boldsymbol{a}$. Then $\boldsymbol{b}\in\boldsymbol{L}^s_\sigma(\Omega)$. Since the space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\boldsymbol{L}^s_\sigma(\Omega)$, there exists a sequence $\boldsymbol{b}_\lambda\in\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ such that $\boldsymbol{b}_\lambda$ converges to $\boldsymbol{b}$ in $\boldsymbol{L}^s(\Omega)$ as $\lambda\to 0$. Therefore, we search for $(\boldsymbol{u}_\lambda,\,\pi_\lambda,\,\boldsymbol{\alpha}_\lambda)\in\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)\times\mathbb{R}^I$ solution of the problem:

$$(OS_N)_\lambda \qquad \begin{cases} -\Delta\, \boldsymbol{u}_\lambda + \boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda + \nabla\, \pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_\lambda = 0 & \text{in } \Omega, \\ \boldsymbol{u}_\lambda \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi_\lambda = \pi_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \pi_\lambda = \pi_0 + \alpha_i^\lambda,\ i = 1, \dots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u}_\lambda \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \dots, I, \end{cases}$$

with $\boldsymbol{\alpha}_\lambda = (\alpha_1^\lambda, \dots, \alpha_I^\lambda)$. Remember that from above we can obtain a unique solution $(\boldsymbol{u}_\lambda,\, \pi_\lambda)$ belonging to $\boldsymbol{W}^{2,6/5}(\Omega) \times W^{1,6/5}(\Omega)$. Since $\boldsymbol{u}_\lambda \in \boldsymbol{L}^6(\Omega)$, then $\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda \in \boldsymbol{L}^6(\Omega)$. Using the Stokes regularity, we deduce that $(\boldsymbol{u}_\lambda,\, \pi_\lambda)$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ if $p \leq 6$. Now, we suppose that $p > 6$. We know that $\boldsymbol{u}_\lambda \in \boldsymbol{W}^{2,6}(\Omega)$ and then $\boldsymbol{u}_\lambda \in \boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda \in \boldsymbol{L}^\infty(\Omega)$. Again using the Stokes regularity, we deduce that $(\boldsymbol{u}_\lambda,\, \pi_\lambda)$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Thus, we focus on the obtention of a strong estimate for $(\boldsymbol{u}_\lambda,\, \pi_\lambda)$ independently from λ. Let $\varepsilon > 0$ with $0 < \lambda < \varepsilon/2$. We consider:

$$\boldsymbol{b}_\lambda = \boldsymbol{b}_1^\varepsilon + \boldsymbol{b}_{\lambda,\,2}^\varepsilon \quad \text{where} \quad \boldsymbol{b}_1^\varepsilon = \widetilde{\boldsymbol{b}} \star \rho_{\varepsilon/2} \quad \text{and} \quad \boldsymbol{b}_{\lambda,\,2}^\varepsilon = \boldsymbol{b}_\lambda - \boldsymbol{b}_1^\varepsilon, \tag{37}$$

being $\widetilde{\boldsymbol{b}}$ the extension by zero of $\boldsymbol{b}$ to $\mathbb{R}^3$ and $\rho_{\varepsilon/2}$ the classical mollifier. By regularity estimates for the Stokes problem, we have

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}} + \|\pi_\lambda\|_{W^{1,p}} \leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi_0\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \sum_{i=1}^{I} |\alpha_i^\lambda| + \|\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}\Big) \tag{38}$$

where the constant C is independent of λ. We note that

$$\begin{aligned} \sum_{i=1}^{I} |\alpha_i^\lambda| &\leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)}(1 + \|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{3/2}(\Omega)}) + \|\pi_0\|_{H^{-1/2}(\Gamma)}\Big) \\ &\leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)}(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}) + \|\pi_0\|_{H^{-1/2}(\Gamma)}\Big). \end{aligned} \tag{39}$$

Now, we use the decomposition (37) in order to bound the term $\|\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}$. We observe first that

$$\|\boldsymbol{b}_{\lambda,\,2}^\varepsilon\|_{\boldsymbol{L}^s(\Omega)} \leq \|\boldsymbol{b}_\lambda - \boldsymbol{b}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{b} - \widetilde{\boldsymbol{b}} \star \rho_{\varepsilon/2}\|_{\boldsymbol{L}^s(\Omega)} \leq \lambda + \varepsilon/2 \leq \varepsilon. \tag{40}$$

Recall that

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^m(\Omega)$$

with $\frac{1}{m} = \frac{1}{p} - \frac{2}{3}$ if $p < \frac{3}{2}$, for any $m \geq 1$ if $p = \frac{3}{2}$ and for any $m \in [1,\, \infty]$ if $p > \frac{3}{2}$. Moreover the imbedding

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^q(\Omega)$$

is compact for any $q < m$ if $p < \frac{3}{2}$, for any $q \in [1,\, \infty[$ if $p = \frac{3}{2}$ and for any $q \in [1,\, \infty]$ if $p > \frac{3}{2}$. Then, using the Hölder inequality and the Sobolev imbedding, we obtain

$$\|\boldsymbol{b}_{\lambda,\,2}^\varepsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{b}_{\lambda,\,2}^\varepsilon\|_{\boldsymbol{L}^s(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^m(\Omega)} \leq C\varepsilon \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}, \tag{41}$$

where $\frac{1}{m} = \frac{1}{p} - \frac{1}{s}$, which is well defined because the definition of the real number s. For the second estimate, we consider two cases.

i) Case $p \leq 3/2$. Let $r \in [\frac{3}{2},\, \infty]$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{6}$ and $t \geq 1$ such that $1 + \frac{1}{r} = \frac{2}{3} + \frac{1}{t}$ satisfying:

$$\begin{aligned} \|\boldsymbol{b}_1^\varepsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} &\leq \|\boldsymbol{b}_1^\varepsilon\|_{\boldsymbol{L}^r(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)} \\ &\leq \|\boldsymbol{b}\|_{\boldsymbol{L}^{3/2}(\Omega)} \|\rho_{\varepsilon/2}\|_{\boldsymbol{L}^t(\mathbb{R}^3)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)}. \end{aligned}$$

Using the estimate (25), we have

$$\|\boldsymbol{b}_1^\varepsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq C_\varepsilon \|\boldsymbol{b}\|_{\boldsymbol{L}^{3/2}(\Omega)} \Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)} \Big). \tag{42}$$

Thanks to the following imbeddings

$$\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^{6/5}(\Omega), \qquad W^{1-1/p,p}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma),$$

we obtain that

$$\|\boldsymbol{b}_1^\varepsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq C_\varepsilon \|\boldsymbol{b}\|_{\boldsymbol{L}^{3/2}(\Omega)} \Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)} \Big). \tag{43}$$

Using (41) and (43), we deduce from (38) and (39) that

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi_\lambda\|_{W^{1,p}(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}) + \|\pi_0\|_{W^{1-\frac{1}{p},p}(\Gamma)} \Big), \tag{44}$$

where in this case $s = \frac{3}{2}$.

ii) Case $p > 3/2$. We know that for any ε' there exists $C'_\varepsilon > 0$ such that

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^\infty(\Omega)} \leq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}_\lambda\|_{\mathbf{L}^6(\Omega)}.$$

Moreover, we have

$$\|\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^\infty(\Omega)}. \tag{45}$$

Thanks to (38), (39) and (25), we deduce the estimate (44) where we replace $\boldsymbol{L}^{3/2}(\Omega)$ by $\boldsymbol{L}^p(\Omega)$ because in this case $s = p$. The estimate (44) is uniform on λ, and therefore we can extract subsequences, that we still call $(\boldsymbol{u}_\lambda)_\lambda$, $(\pi_\lambda)_\lambda$ and $(\alpha_i^\lambda)_\lambda$, such that if $\lambda \to 0$,

$$\boldsymbol{u}_\lambda \longrightarrow \boldsymbol{u} \quad \text{weakly in } \boldsymbol{W}^{2,p}(\Omega),$$

and

$$\pi_\lambda \longrightarrow \pi \quad \text{weakly in } W^{1,p}(\Omega), \qquad \alpha_i^\lambda \longrightarrow \alpha_i \ \text{ for any } i = 1, \ldots, I.$$

It is easy to verify that $(\boldsymbol{u}, \pi, \boldsymbol{\alpha})$ is solution of the problem $(\mathcal{OS}_N)$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$, $\pi = \pi_0$ on Γ_0 and $\pi = \pi_0 + \alpha_i$ on Γ_i for any $i = 1, \ldots I$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies the estimate (36). □

Remark 4. Observe that to obtain the regularity, the value of p cannot be equal to $\frac{3}{2}$ because of $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ and thus $\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u}$ can not be better than $\boldsymbol{L}^{3-\varepsilon}(\Omega)$ $(\varepsilon > 0)$. If we consider the case $p = \frac{3}{2}$ (respectively $p > \frac{3}{2}$), we must suppose that $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$ for arbitrary $\varepsilon > 0$ (respectively $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^p(\Omega)$).

We are now interested in the study of generalized solutions of the problem (29) where $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ for $1 < p < \infty$. As for the case $p = 2$, we choose $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. Observe that the pressure π is a solution of:

$$\begin{cases} \Delta\,\pi = \operatorname{div} \boldsymbol{f} - \operatorname{div}(\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u}) + \Delta\,\chi & \text{in}\,\Omega, \\ \pi = \pi_0 \ \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + \alpha_i & \text{on } \Gamma_i. \end{cases}$$

If $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, then $\boldsymbol{u} \in \boldsymbol{L}^{p*}(\Omega)$ with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$ if $p < 3$, $p*$ arbitrary in $[1, \infty[$ if $p = 3$ and $p* = \infty$ if $p > 3$. Consequently, if we keep the same hypothesis on $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ we have $\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^t(\Omega)$ with $t = r$ if $p < 3$, $t < r = \frac{3}{2}$ if $p = 3$ and $t = \frac{3}{2} < r$ if $p > 3$. Consequently, if $p < 3$, we have $\Delta\,\pi \in W^{-1,r}(\Omega)$, so that if $\pi_0 \in W^{1-1/r,r}(\Gamma)$, then $\pi \in W^{1,r}(\Omega)$. But, if $p \geq 3$, $\Delta\,\pi \notin \boldsymbol{W}^{-1,r}(\Omega)$ if we suppose only that $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$. The next theorem gives the existence of solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ with $p > 2$ provided that $\mathbf{curl}\, \boldsymbol{a}$ is in a space $\boldsymbol{L}^s(\Omega)$ with s large enough.

Theorem 3.6. *Let $p > 2$. Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\chi \in W^{1,r}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. We suppose that $\pi_0 \in W^{1-1/r,r}(\Gamma)$ and $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ and s satisfies:*

$$s = \frac{3}{2} \text{ if } 2 < p < 3, \quad s = \frac{3}{2} + \varepsilon \text{ if } p = 3 \text{ and } s = r \text{ if } p > 3,$$

for some arbitrary $\varepsilon > 0$. Then the problem (29) has a unique solution $(\boldsymbol{u}, \pi, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)} \quad &\leq \quad C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2 \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'} + \\ &+ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)}\Big) \quad (46) \end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$.

Proof. As for the Stokes problem $(\mathcal{S}'_N)$, where the external forces belong to the dual space $\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)$, we can prove that the problem:

$$\begin{cases} -\Delta\, \boldsymbol{u}_0 + \nabla\, q_0 = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u}_0 = \chi & \text{in } \Omega \\ \boldsymbol{u}_0 \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ q_0 = 0 \text{ on } \Gamma_0 \quad \text{ and } \; q_0 = \alpha_i & \text{on } \Gamma_i,\ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u}_0 \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{47}$$

with

$$\alpha_i = \langle \boldsymbol{f},\, \nabla\, q_i^N \rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]' \times \boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)} + \int_\Gamma \chi\, \nabla\, q_i^N \cdot \boldsymbol{n},$$

has a unique solution $(\boldsymbol{u}_0,\, q_0) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ satisfying the estimate

$$\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|q_0\|_{W^{1,r}(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)}\Big). \tag{48}$$

Now, observe that with the values chosen for s and r, since $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p*}(\Omega)$ and $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$, we can verify that $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_0 \in \boldsymbol{L}^r(\Omega)$. Indeed, if $p < 3$, then $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$ and $\frac{1}{s} + \frac{1}{p*} = \frac{1}{r}$. If $p = 3$, then there exists $\varepsilon > 0$ such that $\dfrac{1}{\frac{3}{2} + \varepsilon} + \frac{1}{p*} = \frac{2}{3}$ and if $p > 3$, then $p* = \infty$ and $r = s$. We note also that with this choice of r, we have the imbedding $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ and $r > 6/5$ because $p > 2$. Finally, from Theorem 3.5, we deduce the existence of a unique $(\boldsymbol{z},\, \theta) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$ solution of

$$\begin{cases} -\Delta\, \boldsymbol{z} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{z} + \nabla\, \theta = -\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_0, \quad \operatorname{div} \boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \theta = \pi_0 \text{ on } \Gamma_0 \quad \text{ and } \; \theta = \pi_0 + \beta_i & \text{on } \Gamma_i,\ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{z} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{49}$$

with

$$\beta_i = -\int_\Omega \mathbf{curl}\, \boldsymbol{a} \times (\boldsymbol{u}_0 + \boldsymbol{z}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \int_\Gamma \pi_0\, \nabla\, q_i^N \cdot \boldsymbol{n}.$$

Using (36) we have the estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\theta\|_{W^{1,r}(\Omega)} \leq C\big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\Big(\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\|_{W^{1-1/p,p}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)}\big) + \|\pi_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\Big) \qquad (50)$$

The pair $(\boldsymbol{u},\,\pi) = (\boldsymbol{z} + \boldsymbol{u}_0,\, q_0 + \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ is a solution of (29). Estimate (46) follows from (50) and (48). □

Now, we will treat the case $3/2 \leq p < 2$ and first we consider the case where: $\chi = 0$ and $\boldsymbol{g} = \mathbf{0}$. The case $1 < p < 3/2$ can also be treated, but with more technical difficulties.

We introduce the space:

$$\begin{aligned}\boldsymbol{V}(\Omega) \;=\; &\Big\{(\boldsymbol{v},\,\theta) \in \boldsymbol{W}^{1,p'}(\Omega) \times W^{1,p^{*\prime}}(\Omega),\;\; \operatorname{div}\boldsymbol{v} \in W_0^{1,p^{*\prime}}(\Omega),\\ -\;&\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{v} + \nabla\theta \in [\boldsymbol{H}_0^{p^*,\,p}(\mathbf{curl},\,\Omega)]',\, \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \;\text{ on } \Gamma,\\ &\int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n} = 0,\;\; 1 \leq i \leq I \;\text{ and }\; \theta = 0 \;\text{ on } \Gamma_0,\;\; \theta = \text{constant} \;\text{ on } \Gamma_i\Big\}.\end{aligned}$$

Proposition 3.7. *We suppose that* $\frac{3}{2} \leq p < 2$. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ *and* $\pi_0 \in W^{1-1/r,r}(\Gamma)$ *with*

$$r = \frac{9+6\varepsilon}{9+2\varepsilon} \;\text{ if }\; p = \frac{3}{2} \quad\text{and}\quad r = \frac{3p}{3+p} \;\text{ if }\; \frac{3}{2} < p < 2, \qquad (51)$$

$$s = \frac{3}{2} + \varepsilon \;\text{ if }\; p = \frac{3}{2} \;\text{ and }\; s = \frac{3}{2} \;\text{ if }\; \frac{3}{2} < p < 2, \qquad (52)$$

where $\varepsilon,\,\varepsilon' > 0$ *are arbitrary. Then the problems:*

$$\textit{Find } (\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I \textit{ satisfying } (\mathcal{OS}_N) \qquad (53)$$

and the variational formulation: Find $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ *with* $\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ and $\int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n} = 0$ *such that:*

$$\begin{cases}\forall(\boldsymbol{w},\,\theta) \in \boldsymbol{V}(\Omega) \;\text{ and }\; \boldsymbol{d} \in \mathbb{R}^I,\\ \langle\boldsymbol{u},\, -\Delta\,\boldsymbol{w} - \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w} + \nabla\,\theta\rangle_{\Omega_{p^*,\,p}} - \int_\Omega \pi\operatorname{div}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} = \langle\boldsymbol{f},\,\boldsymbol{w}\rangle_{\Omega_{r',\,p'}} - \int_\Gamma \pi_0\,\boldsymbol{w}\cdot\boldsymbol{n},\\ \forall i = 1,\ldots,I, \quad \alpha_i = \langle\boldsymbol{f},\,\nabla\,q_i^N\rangle_{\Omega_{r',\,p'}} - \int_\Omega(\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\nabla\,q_i^N\,\mathrm{d}\boldsymbol{x} - \int_\Gamma \pi_0\,\nabla\,q_i^N\cdot\boldsymbol{n}.\end{cases} \qquad (54)$$

are equivalent, where $\boldsymbol{d} = (d_1,\ldots,\,d_I)$ *and where the brackets* $\langle\cdot,\,\cdot\rangle_{\Omega_{p^*,\,p}}$ *denotes the duality* $[\boldsymbol{H}_0^{p^*,\,p}(\mathbf{curl},\,\Omega)]' \times \boldsymbol{H}_0^{p^*,\,p}(\mathbf{curl},\,\Omega)$ *and* $\langle\cdot,\,\cdot\rangle_{\Omega_{r',\,p'}}$ *denotes the duality* $[\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)]' \times \boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)$.

Proof. **i)** Let $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ be a solution of $(\mathcal{OS}_N)$ and let $(\boldsymbol{w},\,\theta) \in \boldsymbol{V}(\Omega)$. By the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in the space $\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)$ and in $\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)$, we obtain:

$$\begin{aligned}\langle-\Delta\,\boldsymbol{u},\,\boldsymbol{w}\rangle_{\Omega_{r',\,p'}} &= \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\\ &= \langle\boldsymbol{u},\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w}\rangle_{\Omega_{p^*,\,p}}\\ &= \langle\boldsymbol{u},\,-\Delta\,\boldsymbol{w} + \nabla\operatorname{div}\boldsymbol{w}\rangle_{\Omega_{p^*,\,p}}.\end{aligned}$$

Oberve that since $\nabla\,\mathrm{div}\,\boldsymbol{w} \in \boldsymbol{L}^{p^{*'}}(\Omega)$, we have $\nabla\,\mathrm{div}\,\boldsymbol{w} \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]'$. Moreover, it is clear that $\Delta\,\boldsymbol{w} \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]'$. Then, we can write:

$$\langle \boldsymbol{u},\, -\Delta\,\boldsymbol{w} + \nabla\,\mathrm{div}\,\boldsymbol{w}\rangle_{\Omega_{p^*,\,p}} = \langle \boldsymbol{u},\, -\Delta\,\boldsymbol{w}\rangle_{\Omega_{p^*,\,p}}, \tag{55}$$

because $\int_\Omega \boldsymbol{u}\cdot\nabla\,\mathrm{div}\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} = 0$ since $\mathrm{div}\,\boldsymbol{u} = 0$ in Ω and $\mathrm{div}\,\boldsymbol{w} = 0$ on Γ. Next, as $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} + \nabla\,\pi \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$, then

$$\langle \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} + \nabla\,\pi,\, \boldsymbol{w}\rangle_{\Omega_{r',\,p'}} = \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla\,\pi\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x},$$

where the integrals are well defined because $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ with $\frac{1}{s} + \frac{1}{p*} + \frac{1}{p'^*} = 1$ and $\frac{1}{r} + \frac{1}{p'^*} = 1$. Next, it is clear that

$$\int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} = -\int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}.$$

Also, since $\pi \in W^{1,r}(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}^{p'^*,(p^{*'})^*}(\mathrm{div},\,\Omega)$ we have

$$\int_\Omega \nabla\,\pi\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \pi\,\mathrm{div}\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \pi\,\boldsymbol{w}\cdot\boldsymbol{n}. \tag{56}$$

where $\boldsymbol{H}^{p'^*,\,(p^{*'})^*}(\mathrm{div},\,\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^{p'^*}(\Omega),\ \mathrm{div}\,\boldsymbol{v} \in L^{(p^{*'})^*}(\Omega)\}$. In order to establish (56), we just check that $\frac{1}{r^*} + \frac{1}{p'^*} \le 1$ and we use the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{H}^{p'^*,\,(p^{*'})^*}(\mathrm{div},\,\Omega)$ (see [2]). Now, since $\mathrm{div}\,\boldsymbol{u} = 0$ in Ω we have for any $\theta \in W^{1,p^{*'}}(\Omega)$:

$$0 = -\int_\Omega \theta\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{u}\cdot\nabla\,\theta\,\mathrm{d}\boldsymbol{x} - \int_\Gamma \theta\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma}.$$

Here too, the last Green formula is verified. Indeed, since $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$ wich implies that $\boldsymbol{u} \in \boldsymbol{H}^{p^*,\,p}(\mathrm{div},\,\Omega)$. Let us summarize. We have obtained that:

$$\begin{aligned}\langle \boldsymbol{f},\, \boldsymbol{w}\rangle_{\Omega_{r',\,p'}} &= \langle \boldsymbol{u},\, -\Delta\,\boldsymbol{w}\rangle_{\Omega_{p^*,p}} - \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\\ &+ \int_\Gamma \pi_0\,\boldsymbol{w}\cdot\boldsymbol{n} + \int_\Omega \boldsymbol{u}\cdot\nabla\,\theta\,\mathrm{d}\boldsymbol{x} - \int_\Gamma \theta\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma}.\end{aligned}$$

Using the properties of $\boldsymbol{w}$ and θ, we obtain that $(\boldsymbol{u},\,\pi)$ satisfies the first equation in (54). With the same arguments used in Theorem 3.2, we can prove that the constants α_i, for $1 \le i \le I$ satisfy the last relation in (54).

ii) Reciprocally, let $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$ a solution of (54). First choose $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}(\Omega)$ and $\theta = 0$, then we deduce that

$$-\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{in } \Omega.$$

Next, we choose $\boldsymbol{w} = \mathbf{0}$ and $\theta \in \mathcal{D}(\Omega)$. Then, we obtain that $\mathrm{div}\,\boldsymbol{u} = 0$ in Ω. We know that $\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ and that $\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0$. It remains to prove the boundary condition given on the pressure π. Let then $\boldsymbol{w} \in \boldsymbol{W}^{1,p'}(\Omega)$ with $\boldsymbol{w}\times\boldsymbol{n} = \mathbf{0}$ on Γ and such that $\mathrm{div}\,\boldsymbol{w} \in W_0^{1,p^*}(\Omega)$ and let $\theta \in W^{1,p^{*'}}(\Omega)$. We obtain as previously,

$$\begin{aligned}\langle \boldsymbol{f},\, \boldsymbol{w}\rangle_{\Omega_{r',\,p'}} &= \langle \boldsymbol{u},\, -\Delta\, \boldsymbol{w}\rangle_{\Omega_{p^*,\,p}} + \int_\Omega \big(-\,\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w} + \nabla\,\theta\big)\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} - \\ &- \int_\Omega \nabla\,\theta\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \pi\,\boldsymbol{w}\cdot\boldsymbol{n}.\end{aligned}$$

But, we can decompose $\boldsymbol{w}$ as

$$\boldsymbol{w} \;=\; \boldsymbol{w} - \sum_{i=1}^{I} \big(\int_{\Gamma_i} \boldsymbol{w}\cdot\boldsymbol{n}\big)\,\nabla\, q_i^N + \boldsymbol{z}, \tag{57}$$

with $\boldsymbol{z} = \sum_{i=1}^{I} \big(\int_{\Gamma_i} \boldsymbol{w}\cdot\boldsymbol{n}\big)\,\nabla\, q_i^N$. Setting $\boldsymbol{w}_0 = \boldsymbol{w} - \boldsymbol{z}$, then if $-\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w}_0 + \nabla\,\theta \in [\boldsymbol{H}_0^{p^*,\,p}(\mathbf{curl},\,\Omega)]'$ we obtain:

$$\begin{aligned}\langle \boldsymbol{f},\, \boldsymbol{w}_0\rangle_{\Omega_{r',\,p'}} + \langle \boldsymbol{f},\, \boldsymbol{z}\rangle_{\Omega_{r',\,p'}} &= \langle \boldsymbol{u},\, -\Delta\,\boldsymbol{w}_0 - \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w}_0 + \nabla\,\theta\rangle_{\Omega_{p^*,\,p}} \qquad (58)\\ &- \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{z})\cdot\boldsymbol{u} - \int_\Omega \nabla\,\theta\cdot\boldsymbol{u} - \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{w}_0 \\ &+ \int_\Gamma \pi\,\boldsymbol{w}_0\cdot\boldsymbol{n} + \int_\Gamma \pi\,\boldsymbol{z}\cdot\boldsymbol{n}. \qquad (59)\end{aligned}$$

This last relation is valid for $\boldsymbol{z} = \mathbf{0}$ and we deduce from (54) that:

$$-\int_\Omega \nabla\,\theta\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \pi\,\boldsymbol{w}_0\cdot\boldsymbol{n} - \int_\Gamma \pi_0\,\boldsymbol{w}_0\cdot\boldsymbol{n} = 0.$$

Choosing $\theta = 0$ on Γ_0 and $\theta =$ constant on Γ_i for any $1 \le i \le I$, we obtain

$$\int_\Gamma (\pi - \pi_0)\,\boldsymbol{w}_0\cdot\boldsymbol{n} = 0.$$

Now, we return to the relation (58) and we can write:

$$\begin{aligned}\int_\Gamma \pi\,\boldsymbol{z}\cdot\boldsymbol{n} &= \langle \boldsymbol{f},\, \boldsymbol{z}\rangle_{\Omega_{r',\,p'}} + \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{z})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\\ &= \langle \boldsymbol{f},\, \boldsymbol{z}\rangle_{\Omega_{r',\,p'}} - \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

So that, using the decomposition (57) and the last relation in (54) we obtain that:

$$\begin{aligned}\int_\Gamma \pi\,\boldsymbol{w}\cdot\boldsymbol{n} &= \int_\Gamma \pi\,\boldsymbol{w}_0\cdot\boldsymbol{n} + \int_\Gamma \pi\,\boldsymbol{z}\cdot\boldsymbol{n}\\ &= \int_\Gamma \pi_0\,\boldsymbol{w}_0\cdot\boldsymbol{n} + \langle \boldsymbol{f},\, \boldsymbol{z}\rangle_{\Omega_{r',\,p'}} - \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Gamma \pi_0\,\boldsymbol{w}_0\cdot\boldsymbol{n} + \sum_{i=1}^{I}\big(\int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}\big)\langle \boldsymbol{f},\;\nabla\, q_i^N\rangle_{\Omega_{r',\,p'}}\\ &- \sum_{i=1}^{I}\big(\int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}\big)\int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\,\nabla\, q_i^N\,\mathrm{d}\boldsymbol{x}. \qquad (60)\end{aligned}$$

This implies that

$$\int_\Gamma \pi\,\boldsymbol{w}\cdot\boldsymbol{n} \;=\; \int_\Gamma \pi_0\,\boldsymbol{w}_0\cdot\boldsymbol{n} + \sum_{i=1}^{I}\Big(\int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}\Big)\Big(\int_\Gamma \pi_0\,\nabla\, q_i^N\cdot\boldsymbol{n} + \alpha_i\Big),$$

Using the decomposition (57), we have

$$\int_\Gamma \pi_0\, \boldsymbol{w}_0 \cdot \boldsymbol{n} + \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{w} \cdot \boldsymbol{n} \Big) \int_\Gamma \pi_0\, \nabla\, q_i^N \cdot \boldsymbol{n} = \int_\Gamma \pi_0\, \boldsymbol{w} \cdot \boldsymbol{n},$$

As a consequence

$$\begin{aligned} \int_\Gamma \pi\, \boldsymbol{w} \cdot \boldsymbol{n} &= \int_\Gamma \pi_0\, \boldsymbol{w} \cdot \boldsymbol{n} + \sum_{i=1}^{I} \alpha_i \int_{\Gamma_i} \boldsymbol{w} \cdot \boldsymbol{n} \\ &= \int_\Gamma (\pi_0 + C)\, \boldsymbol{w} \cdot \boldsymbol{n}, \end{aligned}$$

with $C = 0$ on Γ_0 and $C = \alpha_i$ on Γ_i. We use exactly the same arguments as in Theorem 3.2 and we obtain that:

$$\pi = \pi_0 \text{ on } \Gamma_0 \text{ and } \pi = \pi_0 + \alpha_i \text{ on } \Gamma_i,$$

which finishes the proof. □

Now, we are going to solve the problem (54).

Theorem 3.8. *Under the assumptions of Proposition 3.7, the problem (54) has a unique solution $(\boldsymbol{u}, \pi, \boldsymbol{\alpha})$ satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^2 (\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Omega)}), \quad (61)$$

$$\|\pi\|_{W^{1,r}(\Omega)} \leq C(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^3 (\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Omega)}) \quad (62)$$

Proof. **i)** Since $p' > 2$, we know due to Theorem 3.6 that for any $\boldsymbol{F}$ belonging to $[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\, \Omega)]'$ and φ belonging to $W_0^{1,p^{*'}}(\Omega)$, there exists a unique $(\boldsymbol{w},\, \theta,\, \boldsymbol{d}) \in \boldsymbol{W}^{1,p'}(\Omega) \times W^{1,p^{*'}}(\Omega) \times \mathbb{R}^I$ with $\operatorname{div} \boldsymbol{w} \in W_0^{1,p^{*'}}(\Omega)$ such that:

$$\begin{cases} -\Delta \boldsymbol{w} - \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w} + \nabla\theta = \boldsymbol{F}, \quad \operatorname{div} \boldsymbol{w} = \varphi \ \text{ in } \Omega, \\ \boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \\ \theta = 0 \ \text{ on } \Gamma_0 \ \text{ and } \ \theta = d_i \ \text{ on } \Gamma_i,\ 1 \leq i \leq I, \\ \int_{\Gamma_i} \boldsymbol{w} \cdot \mathbf{n} = 0,\ 1 \leq i \leq I, \end{cases} \quad (63)$$

satisfying the estimate:

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\theta\|_{W^{1,p^{*'}}(\Omega)} \leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2 \big(\|\boldsymbol{F}\|_{[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)]'} + \|\varphi\|_{W^{1,p^{*'}}(\Omega)}\big), \quad (64)$$

where $\boldsymbol{d} = (d_1, \ldots, d_I)$ and

$$d_i = \langle \boldsymbol{F}, \nabla\, q_i^N \rangle_\Omega + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x},\ \ 1 \leq i \leq I.$$

Remark that we can apply Theorem 3.6 because the real s defined in (52) satisfies the assumptions of Theoreme 3.6 where however we must replace p by p'. Indeed, since $p' > 2$, we have $s = \frac{3}{2}$ if $2 < p' < 3$, $s = \frac{3}{2} + \varepsilon$ if $p' = 3$. Now, using (64) we obtain:

$$\begin{aligned} &\Big| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\Omega_{r',\,p'}} - \int_\Gamma \pi_0 \boldsymbol{w} \cdot \boldsymbol{n} \Big| \\ \leq\ & \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\pi_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \|\boldsymbol{w}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p'},p'}(\Gamma)} \\ \leq\ & C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\big)(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^2 \times \\ \times\ & \big(\|\boldsymbol{F}\|_{[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]'} + \|\varphi\|_{W^{1,p^{*\prime}}(\Omega)}\big). \end{aligned}$$

In other words, the linear mapping

$$(\mathbf{F}, \varphi) \longrightarrow \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\Omega_{r',\,p'}} - \int_\Gamma \pi_0 \boldsymbol{w}\cdot\boldsymbol{n},$$

defines an element $(\boldsymbol{u},\,\pi)$ of the dual space of $[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]' \times W_0^{1,p^{*\prime}}(\Omega)$ which means that $(\boldsymbol{u},\,\pi) \in \boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega) \times W^{-1,p^*}(\Omega)$ and satisfies:

$$\langle \boldsymbol{u},\,\boldsymbol{F} \rangle_{\Omega_{p^*,\,p}} - \langle \pi,\,\varphi \rangle_{W^{-1,p^*}(\Omega)\times W_0^{1,p^{*\prime}}(\Omega)} = \langle \boldsymbol{f},\,\boldsymbol{w} \rangle_{\Omega_{r',\,p'}} - \int_\Gamma \pi_0 \boldsymbol{w}\cdot\boldsymbol{n}, \tag{65}$$

with

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{H}^{p^*,p}(\mathbf{curl},\,\Omega)} &+ \|\pi\|_{W^{-1,p^*}(\Omega)} \leq C(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^2 \times \\ &\times (\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}). \end{aligned} \tag{66}$$

We have then

$$\begin{aligned} \langle \boldsymbol{u}, -\Delta \boldsymbol{w} - \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w} + \nabla\theta \rangle_{\Omega_{p^*,\,p}} &- \langle \pi, \operatorname{div}\boldsymbol{w} \rangle_{W^{-1,r}(\Omega)\times W_0^{1,r'}(\Omega)} = \\ &= \langle \boldsymbol{f},\,\boldsymbol{w} \rangle_{\Omega_{r',\,p'}} - \int_\Gamma \pi_0 \boldsymbol{w}\cdot\boldsymbol{n} \end{aligned} \tag{67}$$

for any $(\boldsymbol{w},\theta) \in \boldsymbol{W}^{1,p'}(\Omega)\times W^{1,p^{*\prime}}(\Omega)$ with $\operatorname{div}\boldsymbol{w} \in W_0^{1,p^{*\prime}}(\Omega)$ satisfying $\boldsymbol{w}\times\boldsymbol{n} = \mathbf{0}$ on Γ, $\int_{\Gamma_i} \boldsymbol{w}\cdot\boldsymbol{n} = 0$ for any $1 \leq i \leq I$, $-\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w} + \nabla\theta \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]'$, $\theta = 0$ on Γ_0 and finally $\theta =$ constant on Γ_i. Then, we have proved that $(\boldsymbol{u},\,\pi) \in \boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)\times W^{-1,p^*}(\Omega)$ satisfies the first equation in (54)

ii) We prove now that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,r}(\Omega)$ and that the second equation in (54) is verified. We choose $\boldsymbol{w} = \mathbf{0}$ and $\theta \in \mathcal{D}(\Omega)$. Then, we obtain $\operatorname{div}\boldsymbol{u} = 0$ in Ω. Since $\boldsymbol{u} \in \boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)$ and $\operatorname{div}\boldsymbol{u} = 0$ in Ω, we deduce that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. The estimate (61) follows from (66). Next, we choose $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}(\Omega)$ and $\theta \in \mathcal{D}(\Omega)$, we obtain

$$-\Delta\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \qquad \text{in } \Omega. \tag{68}$$

But $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ and $\boldsymbol{u} \in L^{p^*}(\Omega)$ with s defined in (52), we have then $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} \in \boldsymbol{L}^r(\Omega)$ with r defined in (51). So, since $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} + \nabla\pi \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]' \hookrightarrow \boldsymbol{W}^{-1,r^*}(\Omega)$, we have $\nabla\pi \in W^{-1,r^*}(\Omega)$ and then also $\pi \in L^{r^*}(\Omega)$. This means in particular that

$$\langle \pi, \operatorname{div}\boldsymbol{w} \rangle_{W^{-1,p^*}(\Omega)\times W_0^{1,p^{*\prime}}(\Omega)} = \int_\Omega \pi \operatorname{div}\boldsymbol{w},$$

because we have $\operatorname{div}\boldsymbol{w} \in L^{(p^{*\prime})^*}(\Omega) = L^{p'}(\Omega)$ with $\frac{1}{r^*} + \frac{1}{p'} \leq 1$.

It remains to show that $\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0$ and $\pi \in W^{1,r}(\Omega)$. For the first point, we choose in (67) $\boldsymbol{w} = 0$ and $\theta \in W^{1,p^{*\prime}}(\Omega)$ with $\theta = 0$ on Γ_0 and $\theta = \delta_{ij}$ on Γ_j for any $1 \leq j \leq I$ and fixed i in $[1,\, I]$. We deduce from (67) that $\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\,\boldsymbol{\sigma} = 0$ for any i in $[1, I]$. For the second point, we know that π satisfies

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} - \operatorname{div}(\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}) \quad \text{in } \Omega,$$

and as in the point *ii*) in the proof of Theorem 3.2, we can prove that π verifies the boundary conditions:

$$\pi = \pi_0 \quad \text{on } \Gamma_0 \qquad \text{and} \qquad \pi = \pi_0 + \alpha_i \quad \text{on } \Gamma_i.$$

Then, using (51) and (52) we can verify that $\Delta\,\pi \in W^{-1,r}(\Omega)$. So, $\pi \in W^{1,r}(\Omega)$ and satisfies the estimate:

$$\|\pi\|_{W^{1,r}(\Omega)} \leq C\,(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^{p*}(\Omega)} + \|\pi_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}) \tag{69}$$

Finally the estimate (62) follows from (69) and (66). □

The next theorem give an extension of the previous one to the case of the problem (29).

Theorem 3.9. *We suppose that $p < 2$. Let $\boldsymbol{f}$, χ, $\boldsymbol{g}$, π_0 and $\boldsymbol{a}$ such that:*

$$\boldsymbol{f}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]',\ \ \chi\in W^{1,r}(\Omega),\ \ \boldsymbol{g}\in \boldsymbol{W}^{1-1/p,p}(\Gamma),\ \ \pi_0\in W^{1-1/r,r}(\Gamma),$$

and $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ with r and s satisfies (51) and (52). Then the problem (29) has a unique solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$ satisfying the estimates

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \ \leq\ & C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2\Big(\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\\ &+\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\chi\|_{W^{1,r}(\Omega)}\big)+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\Big)\end{aligned} \tag{70}$$

$$\begin{aligned}\|\pi\|_{W^{1,r}(\Omega)} \ \leq\ & C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^3\Big(\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\\ &+\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\chi\|_{W^{1,r}(\Omega)}\big)+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\Big)\end{aligned} \tag{71}$$

where $\boldsymbol{\alpha} = (\alpha_1,\,\ldots,\,\alpha_I)$ with

$$\alpha_i = \langle \boldsymbol{f},\,\nabla\,q_i^N\rangle_{\Omega_{r',\,p'}} + \int_\Gamma(\chi-\pi_0)\,\nabla\,q_i^N\cdot\boldsymbol{n} - \int_\Omega(\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{n})\cdot\nabla q_i^N.$$

Proof. Let $(\boldsymbol{u}_0,\,q_0) \in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)$ such that (47) and satisfies the estimate (48). Next, we consider $(\boldsymbol{z},\,\theta) \in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)$ solution of (49). We note here that we appply Theroem 3.8 because $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)]'$. Indeed, using (51) and (52), we have $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}_0 \in \boldsymbol{L}^r(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)]'$. Moreover using (61), we obtain that $\boldsymbol{z}$ satisfies the estimate:

$$\begin{aligned}\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}} \ &\leq\ C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s}\big)^2\big(\|\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)]'}+\|\pi_0\|_{W^{1-1/r,r}}\big)\\ &\leq\ C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s}\big)^2\big(\|\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}_0\|_{\boldsymbol{L}^r(\Omega)}+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big)\\ &\leq\ C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s}\big)^2\big(\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi_0\|_{W^{1-1/r,r}}\big).\end{aligned}$$

Using the estimate (48), we deduce that $\boldsymbol{z}$ satisfies:

$$\begin{aligned}\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} \ \leq\ & C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2\Big(\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\\ &+\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\chi\|_{W^{1,r}(\Omega)}\big)+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\Big).\end{aligned} \tag{72}$$

Similarly, using (62) and (48), we obatin the estimate:

$$\begin{aligned}\|\theta\|_{W^{1,r}(\Omega)} &\leq C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^3\Big(\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\\ &+ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\chi\|_{W^{1,r}(\Omega)}\big)+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\Big). \qquad (73)\end{aligned}$$

Finally, the pair $(\boldsymbol{u},\,\pi)=(\boldsymbol{z}+\boldsymbol{u}_0,\,q_0+\theta)\in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)$ is a solution of (29). Estimates (70) and (71) follows from (48), (72) and (73). □

4. **The Navier-Stokes problem.** As a consequence of the previous study in the previous sections, we want in this section to study the following Navier-Stokes equations:

$$(\mathcal{NS}_N)\qquad \begin{cases}-\Delta\,\boldsymbol{u}+\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u}+\nabla\,\pi=\boldsymbol{f} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u}=\chi & \text{in } \Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{g} & \text{on } \Gamma,\\ \pi=\pi_0 \ \text{ on } \Gamma_i \text{ and } \pi=\pi_0+c_i & \text{on } \Gamma_i,\ i=1,\ldots,I,\\ \int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma}=0,\ i=1,\ldots,I, & \end{cases}$$

where $\boldsymbol{f},\chi,\boldsymbol{g}$, and π_0 are given functions, c_i are constants and we will denote by $\boldsymbol{c}$ the vector $\boldsymbol{c}=(c_1,\ldots,c_I)$.

In the search of a proof of the existence of generalized solution for the Navier-Stokes equations $(\mathcal{NS}_N)$, we consider the case of small enough data.

Theorem 4.1. *Let* $\boldsymbol{f}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$, $\chi\in W^{1,r}(\Omega)$, $\boldsymbol{g}\in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, $\pi_0\in W^{1-1/r,r}(\Gamma)$ *with* $\frac{3}{2}<p<3$ *and* $r=\frac{3p}{3+p}$.
i) There exists a constant $\alpha_1>0$ *such that, if*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}+\|\chi\|_{W^{1,r}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\leq\alpha_1,$$

then, there exists a solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{c})\in\boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$ *to problem* $(\mathcal{NS}_N)$ *verifying the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}+\|\chi\|_{W^{1,r}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big), \qquad (74)$$

with $c_i=\langle\boldsymbol{f},\,\nabla\,q_i\rangle_{\Omega_{r',\,p'}}+\int_\Gamma(\chi-\pi_0)\,\nabla\,q_i^N\cdot\boldsymbol{n}-\int_\Omega(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\nabla q_i^N$.
ii) Moreover, there exists a constant $\alpha_2\in]0,\alpha_1]$ *such that this solution is unique, if*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}+\|\chi\|_{W^{1,r}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\leq\alpha_2.$$

Proof. i)**Existence:** We begin to prove existence of generalized solutions. We want to apply Banach's fixed point theorem. The idea is to do this fixed point over the Oseen problem (29). We are searching for a fixed point for the mapping T,

$$\begin{aligned}T\ :\ \boldsymbol{W}^{1,p}(\Omega) &\longrightarrow \boldsymbol{W}^{1,p}(\Omega)\\ \boldsymbol{a} &\longmapsto \boldsymbol{u},\end{aligned}$$

where given $\boldsymbol{a}\in\boldsymbol{W}^{1,p}(\Omega)$, $T\,\boldsymbol{a}=\boldsymbol{u}$ is the unique solution of (29), where we replace α_i by c_i, given by Theorem 3.6 and Theorem 3.9. In order to apply the fixed point theorem, we have to define a neighborhood $\boldsymbol{B}_\lambda$, in the form:

$$\boldsymbol{B}_\lambda=\big\{\boldsymbol{a}\in\boldsymbol{W}^{1,p}(\Omega),\quad\|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq\lambda\big\}.$$

Let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2 \in \boldsymbol{B}_\lambda$. If we choose a contraction method, we must prove that: there exists a constant $\theta \in]0, 1[$ such that

$$\|T\boldsymbol{a}_1 - T\boldsymbol{a}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} \le \theta \|\boldsymbol{a}_1 - \boldsymbol{a}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{75}$$

In order to estimate $\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}$, we observe that for each $k = 1, 2$, $(\boldsymbol{u}_k,\, \pi_k) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ verifies:

$$\begin{cases} -\Delta\, \boldsymbol{u}_k + \mathbf{curl}\, \boldsymbol{a}_k \times \boldsymbol{u}_k + \nabla\, \pi_k = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_k = \chi & \text{in } \Omega, \\ \boldsymbol{u}_k \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi_k = \pi_0 \text{ on } \Gamma_0 \text{ and } \pi_k = \pi_0 + c_i & \text{on } \Gamma_i,\ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u}_k \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases}$$

with the estimate:

$$\begin{aligned} \|\boldsymbol{u}_k\|_{\boldsymbol{W}^{1,p}(\Omega)} \quad &\le \quad C(1 + \|\mathbf{curl}\, \boldsymbol{a}_k\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \\ &+ \quad (1 + \|\mathbf{curl}\, \boldsymbol{a}_k\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})(\|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)})\Big). \end{aligned}$$

However, in order to estimate the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we have to reason differently. We start with the problem verified by $(\boldsymbol{u},\, \pi) = (\boldsymbol{u}_1 - \boldsymbol{u}_2,\, \pi_1 - \pi_2)$, which is the following one:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a}_1 \times \boldsymbol{u} + \nabla\, \pi = -\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_2 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = 0 & \text{on } \Gamma, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases}$$

where $\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2$. Using the estimate made for the Oseen problem successively for $\boldsymbol{u}$ and $\boldsymbol{u}_2$, we obtain that:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C(1 + \|\mathbf{curl}\, \boldsymbol{a}_1\|_{\boldsymbol{L}^{3/2}(\Omega)})(\|\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_2\|_{\boldsymbol{L}^{r}(\Omega)}).$$

Then

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \quad &\le \quad C(1 + \|\mathbf{curl}\, \boldsymbol{a}_1\|_{\boldsymbol{L}^{3/2}(\Omega)})\|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\|\boldsymbol{u}_2\|_{\boldsymbol{L}^{p^*}(\Omega)} \qquad (76) \\ &\le \quad C^2(1 + \|\mathbf{curl}\, \boldsymbol{a}_1\|_{\boldsymbol{L}^{3/2}(\Omega)})(1 + \|\mathbf{curl}\, \boldsymbol{a}_2\|_{\boldsymbol{L}^{3/2}(\Omega)})\|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)} \\ &\times \quad \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \big(1 + \|\mathbf{curl}\, \boldsymbol{a}_2\|_{\boldsymbol{L}^{3}/2(\Omega)}\big) \\ &\times \quad \big(\|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big)\Big). \end{aligned}$$

We set $\alpha = \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}$ and $\beta = \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}$. Then, (76) becomes:

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \quad &\le \quad C_1C^2\|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p}(\Omega)}(1 + C_1\|\boldsymbol{a}_1\|_{\boldsymbol{W}^{1,p}(\Omega)})(1 + C_1\|\boldsymbol{a}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}) \times \\ &\times \quad \big(\alpha + \beta(1 + C_1\|\boldsymbol{a}_2\|_{\boldsymbol{W}^{1,p}(\Omega)})\big), \end{aligned} \tag{77}$$

where $C_1 > 0$ is such that

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega), \qquad \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^{3/2}(\Omega)} \le C_1\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{78}$$

Thus, we can (for instance) obtain estimate (75) if we consider r such that

$$C_1C^2(1 + C_1\lambda)^2\big(\alpha + \beta(1 + C_1\lambda)\big) < 1,$$

that is verified, for example, taking:

$$\lambda = C_1^{-1}\Big(\big(2C_1C^2(\alpha+\beta)\big)^{-1/3}-1\Big) \quad \text{and} \quad \alpha+\beta<(2C_1C^2)^{-1}. \tag{79}$$

Therefore, if (79) is verified, then the fixed point $\boldsymbol{u}^* \in \boldsymbol{W}^{1,p}(\Omega)$ satisfies the estimate:

$$\|\boldsymbol{u}^*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\alpha+\beta)\big(1+C_1\|\boldsymbol{u}^*\|_{\boldsymbol{W}^{1,p}(\Omega)}\big)^2. \tag{80}$$

i.e

$$\|\boldsymbol{u}^*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq x^*,$$

where x^* is the smallest solution of the equation: $x = ax^2+1$ with $a = CC_1(\alpha+\beta)$, *i.e* $x^* = \dfrac{2}{1+\sqrt{1-4a}}$, where we suppose that $a < 1/4$. Thus,

$$\|\boldsymbol{u}^*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \frac{4a}{(1+\sqrt{1-4a})^2} \leq 4a, \tag{81}$$

which proves the point i) and the estimate (74).
ii) **Uniqueness:** We shall next prove uniqueness. Let us denote by $(\boldsymbol{u}_1, \pi_1)$ the solution obtained in step i) and by $(\boldsymbol{u}_2, \pi_2)$ any other solution for $(\mathcal{NS}_N)$ corresponding to the same data. Setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\pi = \pi_1 - \pi_2$. We find that:

$$\begin{cases} -\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{u}_1 \times \boldsymbol{u} + \nabla\,\pi = -\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{u}_2 & \text{in } \Omega, \\ \operatorname{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = 0 & \text{on } \Gamma, \\ \int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1,\ldots,I. \end{cases} \tag{82}$$

Then,

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1+\|\mathbf{curl}\,\boldsymbol{u}_1\|_{\boldsymbol{L}^{3/2}(\Omega)})(\|\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u}_2\|_{\boldsymbol{L}^r(\Omega)}), \tag{83}$$

i.e as in step i):

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq (1+C_1\|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)})C_1(\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{84}$$

But, for $i = 1, 2$

$$\|\boldsymbol{u}_i\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1+C_1\|\boldsymbol{u}_i\|_{\boldsymbol{W}^{1,p}(\Omega)})^2(\alpha+\beta), \tag{85}$$

and then,

$$\|\boldsymbol{u}_i\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq 4\gamma CC_1, \tag{86}$$

where $\gamma = \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}$, we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq CC_1^2\big(1+4C_1^2C\gamma\big)4CC_1\gamma\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{87}$$

Then, if $CC_1^2\big(1+4C_1^2C\gamma\big)4CC_1\gamma < 1$, we deduce that $\boldsymbol{u} = \mathbf{0}$ and the proof of uniqueness is completed. □

REFERENCES

[1] C. Amrouche and V. Girault, *Decomposition of vector space and application to the Stokes problem in arbitrary dimension*, Czechoslovak Math. J., **44** (1994), 109–140.

[2] C. Amrouche and M. Ángeles Rodríguez-Bellido, *Stationary Stokes, Oseen and Navier-Stokes equations with singular data*, Arch. Rational. Mech. Anal., **199** (2011), 597–651.

[3] C. Amrouche and N. Seloula, *L^p-theory for vector potentials and Sobolev's inequalities for vector fields. Application to the Stokes equations with pressure boundary condition*, to appear in M3AS.

[4] C. Amrouche and N. Seloula, *On the Stokes equations with the Navier-type boundary conditions*, Differential Equations and Applications, **3** (2011), 581–607.

[5] C. Conca, F. Murat and O. Pironneau, *The Stokes and Navier-Stokes equations with boundary conditions involving the pressure*, Japan. J. Math. (N.S.), **20** (1994), 263–318.

Received December 2011; revised April 2012.

E-mail address: cherif.amrouche@univ-pau.fr
E-mail address: nour-elhouda.seloula@unicaen.fr